Glasgow Caledonian University

SCHOOL OF ENGINEERING & BUILT ENVIRONMENT

Mathematics

An Introduction to Graph Theory

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An Introduction to Graph Theory

GRAPH THEORY

1. Introduction

In recent years graph theory has become established as an important area of mathematics and computer science. The origins of graph theory can be traced back to Swiss mathematician Euler and his work on the Königsberg bridges problem (1735), shown schematically in Figure 1.

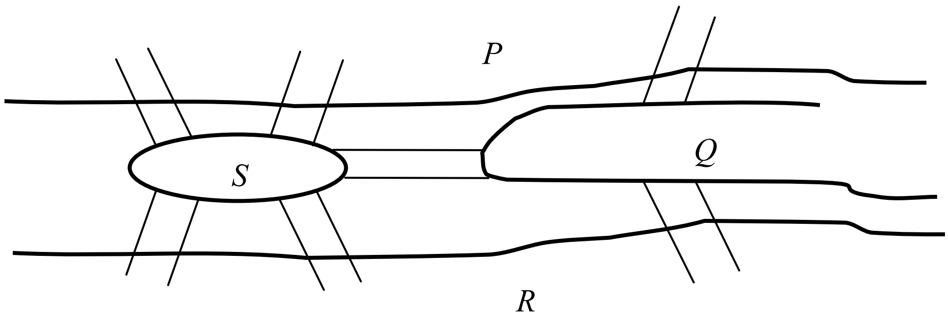


Figure 1: Bridges of Königsberg

Königsberg was a city in Germany (it is now called Kaliningrad and is in western Russia) and the river Pregel, with an island in the middle, ran through it. Seven bridges were built so that the city's inhabitants could travel between the four parts of the city; *P*, *Q*, *R* and *S* in Figure 1. The people wondered whether or not it was possible to walk around the city in such a way that each bridge was crossed exactly once, ending up at the starting point. However, all attempts to do so, including Euler's, ended in failure. Euler reasoned that anyone standing on land would have to have a way to get on and off. Therefore each land mass would need an even number of bridges, or if the journey started at one land mass and ended at another, then only those two land masses could have an odd number of bridges. However, in Königsberg each land mass had an odd number of bridges explaining why all seven bridges could not be crossed without crossing one more than once. In formulating his solution Euler simplified the bridge problem by representing each land mass as a point and each bridge as a line as shown in Figure 2, leading to the introduction of graph theory and the concept of an Eulerian graph.

Another two well known examples from graph theory are:

1. How many colours do we need to colour a map so that every pair of countries with a border in common have different colours?
2. Given a map of several cities and the roads between them, is it possible for a travelling salesman to visit (pass through) each of the cities exactly once?

Some of the applications of graph theory include: communication network design, GPS to find the shortest path between two points, design of electrical circuits and modelling of the Worldwide Web.

2. Definitions

The Königsberg Bridge problem can be represented diagramatically by means of a set of points and lines. The points *P*, *Q*, *R* and *S* are called **vertices**, the lines are called **edges** (or **arcs**) and the whole diagram is called a **graph**.

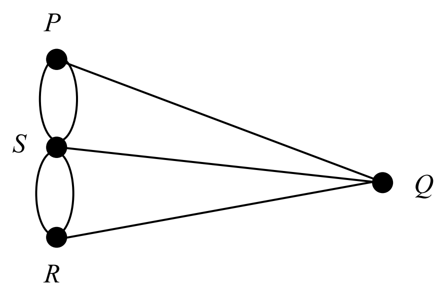


Figure 2: Graph theory representation of the Königsberg bridge problem

A graph, *G*, is a mathematical structure which consists of:

(i). a set $V = V(G)$ whose elements are called vertices, points or nodes of *G*.
(ii). a set $E = E(G)$ of unordered pairs of distinct vertices called edges of *G*.

Such a graph is denoted $G = \{ V(G), E(G) \}$ or $G = \{ V, E \}$.

Example 1

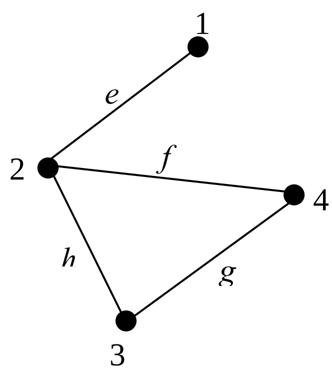


Figure 3: Graph with 4 vertices and 4 edges

The set *V* consists of the vertices, 1, 2, 3 and 4, i.e. $V(G) = \{ 1, 2, 3,4 \}$

The set *E* consists of the edges, $e = (1, 2)$, $f = (2, 4)$, $g = (3, 4)$ and $h = (2, 3)$, i.e. $E(G) = \{ (1, 2), (2, 4), (3, 4), (2, 3) \}$.

Hence, $G = \{ V(G), E(G) \} = \{ \{ 1, 2, 3,4 \}, \{(1, 2), (2, 4), (3, 4), (2, 3)\} \}$.

An **undirected graph** is a graph in which the edges have no orientation. Hence, in an undirected graph the edge set is composed of unordered vertex pairs. In Fig. 3 for example, the edge (1, 2) is considered identical to the edge (2, 1).

If *X* and *Y* are vertices of a graph, *G*, then *X* and *Y* are said to be **adjacent** if they are joined by an edge.
In Fig. 3, 1 and 2 are adjacent, 2 and 4 are adjacent but 1 and 4 are not adjacent.

An edge in a graph that joins two vertices is said to be **incident** to both vertices.
In Fig. 3, edge *e* is incident to vertices1 and 2, *h* is incident to vertices 2 and 3.

In Fig. 3, 1 and 2 are called the **endpoints** of *e* and *e* is said to **connect** 1 and 2.

The **order** of a graph, denoted $|V(G)|$, is the number of vertices contained in *G*.
In Fig. 3, $|V(G)| = 4$.

The **size** of a graph, denoted $|E(G)|$, is the number of edges contained in *G*.
In Fig. 3, $|E(G)| = 4$.

The **degree** of a vertex *X*, written $deg(X)$, is the number of edges to which *X* is incident.
In Fig. 3 deg(1) = 1, deg(2) = 3, deg(3) = 2, deg(4) = 2

Any vertex of degree zero is called an **isolated** vertex and a vertex of degree one is an **end-vertex**.

A vertex is said to be **even** or **odd** according to whether its degree is an even or odd number. In Fig.3 vertices 1 and 2 are odd while vertices 3 and 4 are even.

If the degrees of all the vertices in a graph, *G*, are summed then the result is an even number. Furthermore, this degree is actually twice the number of edges, as each edge contributes 2 to the total sum. We have the following lemma:

Lemma (Handshaking Lemma)
In any graph the sum of the vertex degrees is equal to twice the number of edges, i.e.

$$\sum_{X \in V(G)} \deg(X) = 2|E(G)|$$

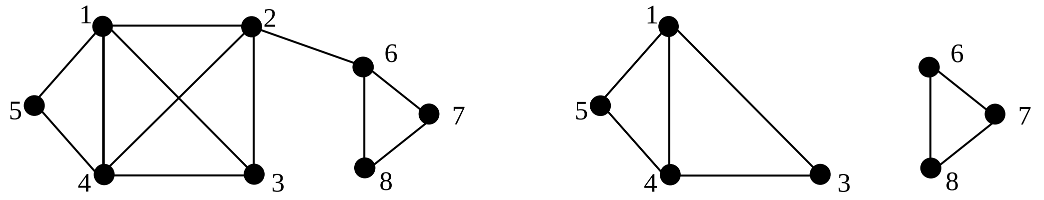
Proof: In a graph *G* an arbitrary edge, *XY*, say, contributes 1 to $deg(X)$ and 1 to $deg(Y)$. Hence the degree sum for the graph is even.

Note: A corollary of the Handshaking Lemma states that the number of odd vertices in a graph must be even.

The **degree sequence** of an undirected graph *G* is a bracketed list of the degrees of all the vertices written in non-decreasing order.
The degree sequence of the graph in Fig. 3 is, (1, 2, 2, 3).

A vertex is a **cut-point** if removal of the vertex disconnects the graph.

Example 2: In graph below vertex 2 is a cut-point as its removal disconnects the graph. The resulting graph has two connected components.



An edge is a **bridge (or isthmus)** if removal of the edge disconnects the graph.

Example 3: Edge (2, 6) is a bridge as its removal disconnects the graph.

3.Graph Structures

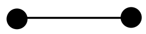
In this section we briefly look at different types of graphs.

3.1. Regular Graphs

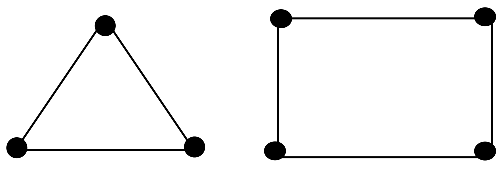
A graph *G* is **regular** if all vertices of *G* have the same degree. A regular graph where all vertices have degree *k* is referred to as a *k*-regular graph.

0-regular:

1-regular:



2-regular



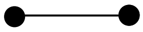
Note: The Handshaking Lemma tells us that the total degree of any graph is an even number, i.e. twice the number of edges. Hence, it is impossible to construct a *k*-regular graph where *k* is odd with an odd number of vertices. For example, we cannot have a 3-regular graph with 5 vertices as this would give a degree sum of 15.

3.2. Complete Graphs

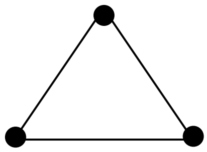
The **complete** graph, denoted K_n, with n vertices, all of which are adjacent to each other, is regular.

K_1

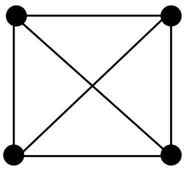
K_2



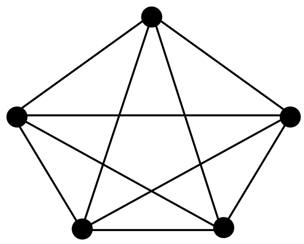
K_3



K_4



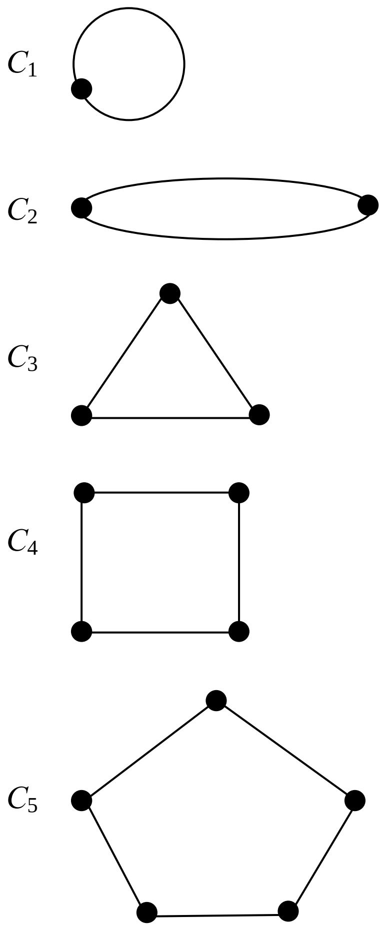
K_5



Note: The complete graph K_n is regular and each of the n vertices has degree $n - 1$. Hence, the sum of the degrees is $n(n - 1)$. Hence, by the Handshaking Lemma the number of edges in K_n is $\frac{n(n-1)}{2}$.

3.3 Cycle Graph

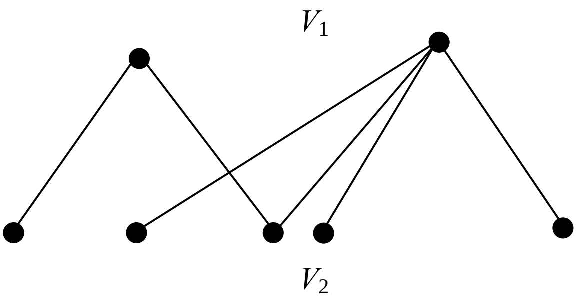
A **cycle graph**, denoted C_n, is a graph on n vertices $\{v_0, v_1, \ldots, v_{n-1}\}$ with n edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_0)$. Note that C_n contains a single **cycle** through all the vertices. See Section 4 for a definition of a cycle in graph theory terms.

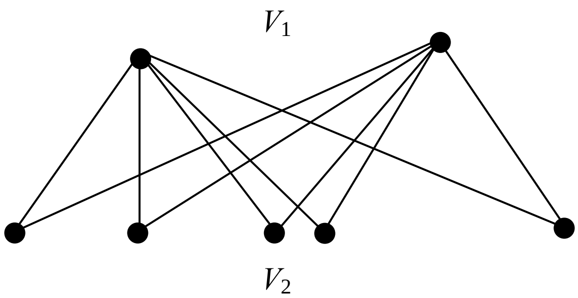


Note: In a cycle graph every vertex has degree 2. We note here that the graph C_1 contains a self-loop and we shall see later in the section on multigraphs that a loop contributes two to the degree of the vertex. Hence, the vertex in C_1 has degree 2.

3.4. Bipartite Graphs

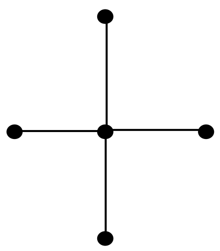
A **bipartite graph**, $G(V_1, V_2)$, is a graph whose vertices can be partitioned into two disjoint subsets V_1 and V_2, where no edge joins vertices that are in the same subset. A vertex in one of the subsets may be joined to all, some, or none of the vertices in the other – see Figures below. In the case where G is simple and every vertex of V_1 is joined to V_2 then G is called a **complete bipartite graph** and is usually denoted $K_{r,s}$ where r and s represent the number of vertices in V_1 and V_2 respectively. A bipartite graph is usually shown with the two subsets as top and bottom rows of vertices or with the two subsets as left and right columns of vertices.



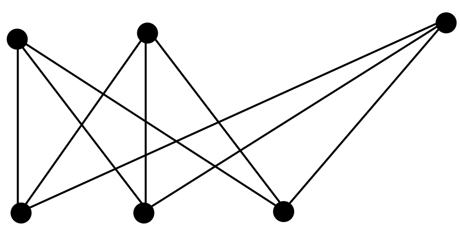


The graph on the right is the complete bipartite graph, $K_{2,5}$ with 5 + 2 = 7 vertices and 5 × 2 = 10 edges. In general, a complete bipartite graph $K_{r,s}$ has $r + s$ vertices and $r \times s$ edges.

A complete bipartite graph of the form $K_{1,s}$ is called a **star graph** and $K_{1,4}$ is shown below.



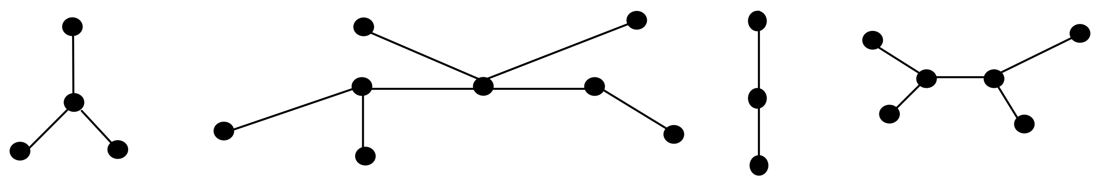
A bipartite graph $K_{r,s}$ is **regular** if and only if $r = s$.



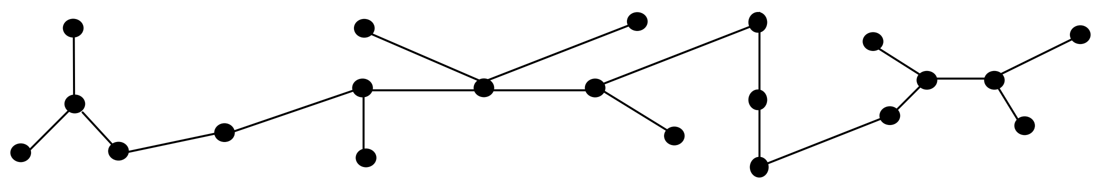
The above complete bipartite graph, $K_{3,3}$ is regular as each vertex has degree 3.

3.5. Tree Graphs:

A **forest** is a graph containing no cycles and a connected forest is called a **tree**. Note that a graph on n vertices has fewest edges when it is a tree (as it has no cycles) and most edges when it is a complete graph. Below is a forest with four components.



If the four components in the above forest are connected we obtain the tree below.



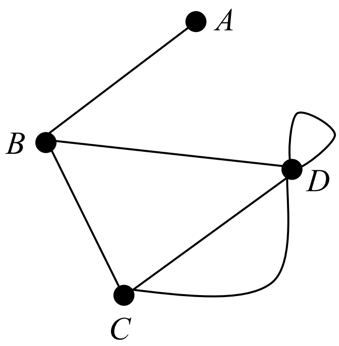
Theorem: Let T be a graph with $n > 1$ vertices. The following statements are equivalent:

(i) T is a tree.
(ii) T is cycle-free and has $n - 1$ edges.
(iii) T is connected and has $n - 1$ edges.
(iv) T is connected and contains no cycles.
(v) T is connected and each edge is a **bridge**.
(vi) any two vertices of T are connected by exactly one path.
(vii) T contains no cycles, but the addition of any new edge creates exactly one cycle.

Note: From the above theorem it must be the case that a finite tree with n vertices must have $n - 1$ edges.

3.6. Multigraphs

Consider the graph G ={ { *A*, *B*, *C*, *D* }, { (*A*,*B*), (*B*,*C*), (*B*,*D*), (*C*,*D*), (*C*,*D*), (*D*,*D*) } }. shown below



This is an example of a **multigraph**. A multigraph is a graph that allows the existence of **loops** and **multiple edges**.

A **loop** is an edge that links a vertex to itself. In the figure the edge (*D*, *D*) is a loop and connects vertex *D* to itself.

If two vertices are joined by more than one edge then these edges are called **multiple edges**. In the figure the edge (*C*, *D*) represents multiple edges.

A **simple** graph is one that contains no loops or multiple edges.

Notes

1. We define a loop to contribute 2 to the degree of a vertex so that the Handshaking Lemma holds for multigraphs. In the above figure vertex *D* therefore has degree 5. The degree sum of the graph is 1 + 3 + 3 + 5 = 12 which is twice the number of edges (6) as required by the Handshaking Lemma.

2. Some texts do not allow multigraphs to have loops.

4. Walks, Trails & Paths

A **walk** of length k on a graph G is an alternating sequence of vertices (v_i) and edges (e_i):

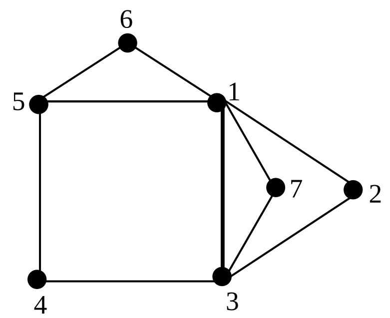
$$v_0, e_1, v_1, e_2, v_2, e_3, \ldots, e_{k-1}, v_k$$

where v_i and v_{i+1} are both incident to e_{i+1}. Note that the graph has $k + 1$ vertices and k edges.

The **length** of a walk is the number of edges in the walk.

For convenience we omit edges and use only vertices so that the walk given above is written as $v_0, v_1, v_2, \ldots, v_k$.

Example 4: A walk on the graph below is given by: 1, 5, 4, 3, 7, 1, 6 and has length L = 6.



A walk can traverse any edge and any vertex any number of times.

A walk is said to be **closed** if its first and last vertices are the same, i.e. $v_0 = v_k$.

Example 5: A **closed walk** is given by: 1, 5, 4, 3, 7, 1, 6, 5, 1.

A **trail** is a walk where all edges are distinct but vertices may be repeated.

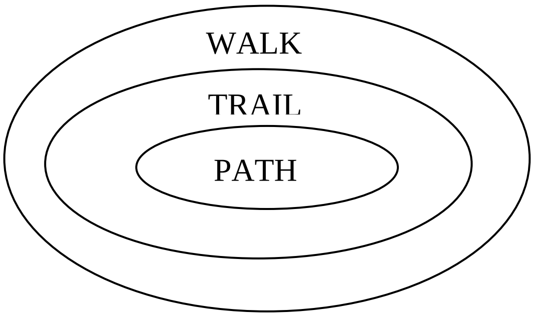
Example 6: A trail is given by: 1, 5, 4, 3, 7, 1, 6, 5.

A **path** is a trail in which all vertices are distinct. Hence, in a path neither vertices nor edges are repeated.

Example 7: A path is given by: 1, 5, 4, 3, 7.

Therefore, all trails are walks and all paths are trails.

In terms of set theory, Paths ⊆ Trails ⊆ Walks as shown below.



A **circuit** is a closed trail.

Example 8: A circuit is given by: 1, 2, 3, 1, 5, 4, 3, 7, 1 is a circuit. Note that no edges are repeated but we are allowed to repeat vertices.

A **cycle** is a closed path

Example 9: A cycle is given by: 1, 2, 3, 4, 5, 1. Note that no vertices (or edges) are repeated.

5. Eulerian and Hamiltonian Graphs

This section considers special ways of traversing graphs. Examples of these traversals are the Königsberg bridges and Travelling Salesman problems.

5.1. Eulerian Graphs

Definition: An **Euler circuit** of a graph, *G* is a path through *G* that starts and ends at the same vertex and uses each edge exactly once. Note that we are allowed to use the same vertex multiple times, but we can only use each edge once. A graph is **Eulerian** if it has an Euler circuit.

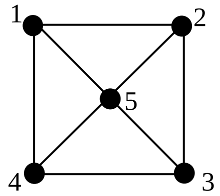
Definition: A **Euler trail** through a graph, *G* is an open trail that passes exactly once through each edge of *G*. We say that *G* is **semi-Eulerian** if it has an Euler trail. Note that every Eulerian graph is semi-Eulerian.

Theorem: Let *G* be a connected graph. Then *G* is Eulerian if and only if every vertex of *G* has even degree.

Corollary: A connected graph is semi-Eulerian if and only if there are 0 or 2 vertices of odd degree. Note that if a semi-Eulerian graph has two vertices of odd degree then any Euler trail must have one of them as its initial vertex and the other as its final vertex.

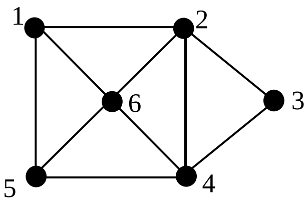
Example 10:

(i)



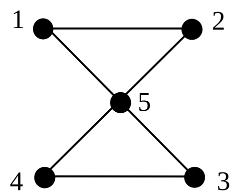
NON-EULERIAN
As there are four vertices of odd degree the graph is non-Eulerian.

(ii)



SEMI-EULERIAN
By the above corollary as there are two vertices of odd degree (i.e. degree 3) then the graph is semi-Eulerian. Euler trail must start at one of the odd degree vertices and end at the other, e.g. 12342645615

(iii)



EULERIAN
(By the above theorem all vertices have even degree and so the graph is Eulerian).
Euler circuit: 1253451

The table below provides simple rules that count the number of odd degree vertices in a graph to decide whether or not it has an Euler circuit or Euler trail.

No. of Odd Vertices	**For a Connected Graph**
0	There is at least one Euler circuit.
1	Not possible
2	No Euler circuit but at least 1 Euler trail.
More than 2	No Euler circuits or Euler trails.

Theorem: If *G* is an Eulerian graph then using the following procedure, known as **Fleury's Algorithm**, it is always possible to construct an Euler circuit of *G*.

Starting at any vertex of *G* traverse the edges of *G* in an arbitrary manner according to the following rules:
- (i) erase edges as they are traversed and if any isolated vertices appear erase them.
- (ii) At each step use a **bridge** only if there is no alternative (see below for a definition of 'bridge').

Note: Since every vertex in the Königsberg graph in Figure 2 has an odd degree it is not possible to find an Euler circuit of this graph. It is therefore impossible for someone to walk around the city in such a way that each bridge is crossed exactly once and end up at the starting point.

5.2. Hamiltonian Graphs

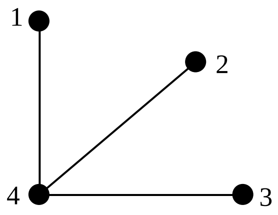
Definition: A circuit (closed trail) which passes exactly once through each vertex of a graph *G* is called a **Hamiltonian circuit** and *G* is called a **Hamiltonian graph**. Note that we do not need to use all the edges.

Definition: A trail that passes exactly once through each vertex of *G* and is not closed is called a **Hamiltonian trail**. We say that *G* is **semi-Hamiltonian**. Note that every Hamiltonian graph is semi-Hamiltonian.

Note that while we have a theorem that provides necessary and sufficient conditions for a connected graph to be Eulerian (i.e. '*G* is Eulerian if and only if every vertex of *G* has even degree') there is no similar characterization for Hamiltonian graphs – this is one of the unsolved problems in graph theory. In general, it is much harder to find a Hamiltonian circuit than it is to find an Eulerian circuit.

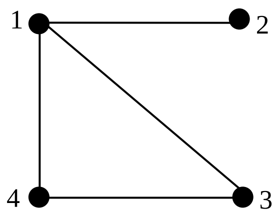
Example 11

(i)



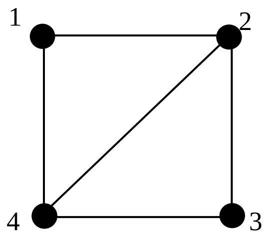
NON-HAMILTONIAN

(ii)



SEMI-HAMILTONIAN
Hamiltonian trail: 2143

(iii)



HAMILTONIAN
Hamiltonian circuit: 12341
Note that we do not need to use all edges.

Note: The Travelling Salesman problem (TSP) searches for the most efficient (least total distance) Hamiltonian circuit a salesman can take so that each of *n* cities is visited. To date, no solution to the TSP has been found.

Note: An Eulerian circuit traverses every edge in a graph exactly once, and may repeat vertices. A Hamiltonian circuit, on the other hand, visits each vertex in a graph exactly once but does not need to use every edge.

6. Graphs and Adjacency Matrices

Up to now we have only considered graphs where the number of edges and vertices is relatively small so that they can be easily be shown in diagram form. However, as graphs become large it is no longer feasible to display them visually. When storing a graph on a computer it is useful to represent it in matrix form, as the calculation of paths, trails and circuits, for example, can easily be performed. If G is a graph with n vertices its **adjacency matrix**, A is defined as the $n \times n$ binary matrix whose ij-th entry is the number of edges joining vertex i and vertex j. In this section we look at how to form the adjacency matrix for different types of graph.

6.1. Undirected Graphs

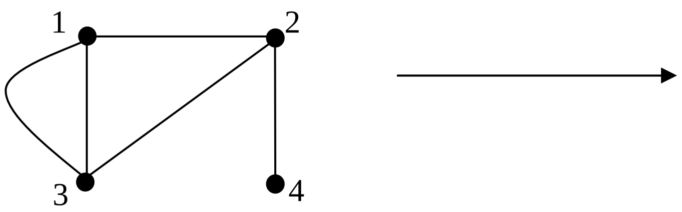
In Section 2 we defined an **undirected graph** to be a graph in which the edges have no orientation. Hence, all edges are bidirectional. For example, in the graph shown in Example 12 below the edge (1, 2) is considered identical to the edge (2, 1).

6.1.1 Adjacency Matrix of an Undirected Graph

The adjacency matrix for an undirected graph is symmetric, i.e. $A = A^T$.

In an undirected multigraph we define a loop to contribute 2 to the degree of a vertex. This approach ensures that the Handshaking Lemma holds for multigraphs.

Example 12



$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Solution

The graph has 4 vertices and so the adjacency matrix will have dimension 4×4.
The entries of the matrix are determined as follows:

- 0 edges connect vertex 1 to vertex 1, so the entry in Row1/Column1 is a '0'
- 1 edge connects vertex 1 to vertex 2, so the entry in Row1/Column2 is a '1'
- 2 edges connect vertex 1 to vertex 3, so the entry in Row1/Column3 is a '2'
- 0 edges connect vertex 1 to vertex 4, so the entry in Row1/Column4 is a '0'

- 1 edge connects vertex 2 to vertex 1, so the entry in Row2/Column1 is a '1'
- 0 edges connect vertex 2 to vertex 2, so the entry in Row2/Column2 is a '0'
- 1 edge connects vertex 2 to vertex 3, so the entry in Row2/Column3 is a '1'
- 1 edge connects vertex 2 to vertex 4, so the entry in Row2/Column4 is a '1'

- 2 edges connect vertex 3 to vertex 1, so the entry in Row3/Column1 is a '2'
- 1 edge connects vertex 3 to vertex 2, so the entry in Row3/Column2 is a '1'
- 0 edges connect vertex 3 to vertex 3, so the entry in Row3/Column3 is a '0'
- 0 edges connect vertex 3 to vertex 4, so the entry in Row3/Column4 is a '0'

- 0 edges connect vertex 4 to vertex 1, so the entry in Row4/Column1 is a ‘0’
- 1 edge connects vertex 4 to vertex 2, so the entry in Row4/Column2 is a ‘1’
- 0 edges connect vertex 4 to vertex 3, so the entry in Row4/Column3 is a ‘0’
- 0 edges connect vertex 4 to vertex 4, so the entry in Row4/Column4 is a ‘0’

Notes

1. For the adjacency matrix of an undirected graph we have that:

Sum of Row j = Sum of Column j = Degree of vertex j

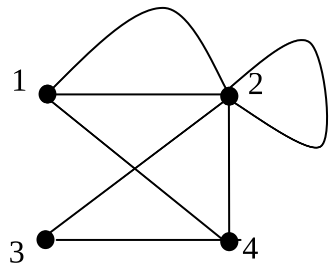
Here,

Sum of Row 1 = Sum of Column 1 = Degree of vertex 1 = 3
Sum of Row 2 = Sum of Column 2 = Degree of vertex 2 = 3
Sum of Row 3 = Sum of Column 3 = Degree of vertex 3 = 3
Sum of Row 4 = Sum of Column 4 = Degree of vertex 4 = 1

2. The degree sum of the graph is 3 + 3 + 3 + 1 = 10 which is twice the number of edges (5) as required by the Handshaking Lemma.

Example 13: Given an adjacency matrix we can construct the associated graph, *G*.

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \longrightarrow$$



The matrix has dimension 4×4 and so the graph has 4 vertices. We proceed as follows:

- Entry in Row1/Column1 is a ‘0’so 0 edges connect vertex 1 to vertex 1
- Entry in Row1/Column2 is a ‘2’so 2 edges connect vertex 1 to vertex 2
- Entry in Row1/Column3 is a ‘0’so 0 edges connect vertex 1 to vertex 3
- Entry in Row1/Column4 is a ‘1’so 1 edge connects vertex 1 to vertex 4

- Entry in Row2/Column1 is a ‘2’so 2 edges connect vertex 2 to vertex 1
- Entry in Row2/Column2 is a ‘2’so vertex 2 has a self-loop
- Entry in Row2/Column3 is a ‘1’so 1 edge connects vertex 2 to vertex 3
- Entry in Row2/Column4 is a ‘1’so 1 edge connects vertex 2 to vertex 4

and so on.

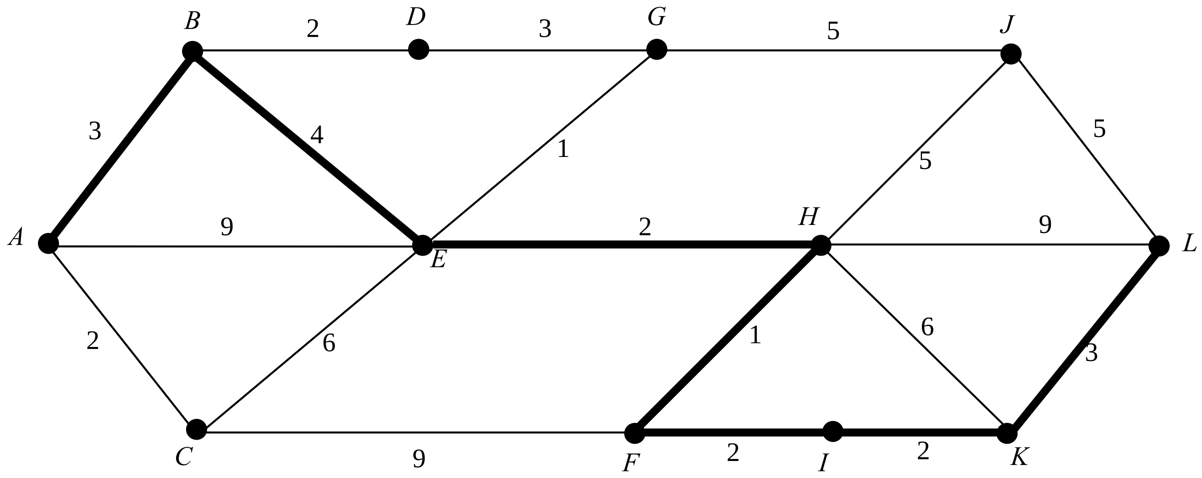
Notes

1. Sum of Row j = Sum of Column j = Degree of vertex j, e.g.
Sum of Row 2 = Sum of Column 2 = Degree of vertex 2 = 6.

2. The degree sum of the graph is 3 + 6 + 2 + 3 = 14 which is twice the number of edges (7) as required by the Handshaking Lemma.

6.2. Weighted Graphs: The edges in a graph can be weighted or unweighted. In a weighted graph a non-negative real number is assigned to each edge, *e*, and is called the **weight** of *e*, denoted *w*(*e*). These weights may correspond to the lengths of roads (edges) between towns (vertices) in a graphical representation of a map and we may be required to find the length of the shortest path from town *A* to town *L* say. The problem is then to find the path from *A* to *L* with minimum weight. An example of a shortest path problem is given by the well-known Travelling Salesman Problem.

Example 12: Find the shortest path from *A* to *L*.



(from Introduction to Graph Theory, Fourth Edition, Wilson R.J., 1996)

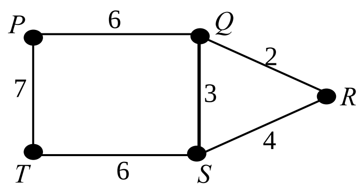
Solution

- Move across the graph from left to right and associate with each vertex *V* a number $l(V)$ giving the shortest distance from *A* to *V*.
- Let vertex *A* have label 0.
- Define temporary labels for *B*, *E*, and *C* as $l(A)+3$, $l(A)+9$ and $l(A)+2$ respectively, i.e. temporary labels are 3, 9 and 2.
- Find the smallest of these. Set $l(C)=2$ so that *C* is now permanently labeled, 2.
- Consider all vertices adjacent to *C*.
  Assign *F* the temporary label, $l(C)+9=11$ and
  Assign *E* the temporary label, $l(C)+6=8$.
  The smallest temporary label is now 3 at *B* and so set $l(B) = 3$.
- Now consider vertices adjacent to *B*.
  Assign *D* the temporary label, $l(B)+2=5$ and
  Assign *E* the temporary label, $l(B)+4=7$.
  The smallest temporary label is now at *D* and so set $l(D) = 5$.
- Continue in this way to get permanent labels: $l(E) = 7$, $l(G) = 8$, $l(H) = 9$, $l(F) = 10$, $l(I) = 12$, $l(J) = 13$, $l(K) = 14$ and $l(L) = 17$.
- The shortest path from *A* to *L* therefore has length 17 and is shown in bold in the above figure.

6.2.1Adjacency Matrix of a Weighted Graph

The adjacency matrix is calculated in the same way as for the previous examples except that instead of placing a 1 in the i^{th} row and j^{th} column when vertices v_i and v_j are adjacent we enter the weight.

Example 14



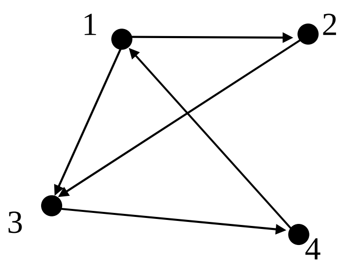
$$A = \begin{bmatrix} 0 & 6 & 0 & 0 & 7 \\ 6 & 0 & 2 & 3 & 0 \\ 0 & 2 & 0 & 4 & 0 \\ 0 & 3 & 4 & 0 & 6 \\ 7 & 0 & 0 & 6 & 0 \end{bmatrix}$$

6.3. Directed Graphs (Digraphs)

The figures above are examples of **undirected** graphs where the direction of an edge is undefined and you can move in both directions between vertices.

In a **directed graph**, or **digraph**, as shown below, the direction of an edge is defined and you can only move between two vertices in that direction. The graph below is represented by

$$G(V, E) = \{ \{1, 2, 3, 4\}, \{(1,2), (1,3), (2,3), (3,4), (4,1)\} \}.$$



The **indegree** of a vertex is the number of edges that terminate at that vertex.

The **outdegree** of a vertex is the number of edges that originate at that vertex.

The edges can be **weighted** or **unweighted** as for undirected graphs.

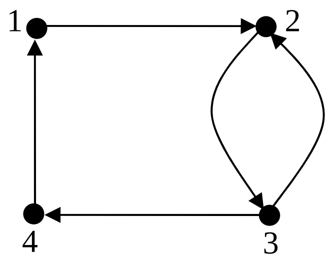
A digraph is **Eulerian** if the indegree equals the outdegree for each vertex.

6.3.1Adjacency Matrix of a Digraph

- The adjacency matrix of a digraph having n vertices is a $n \times n$ binary matrix.
- For each directed edge (v_i, v_j), i.e. arrow from vertex v_i to vertex v_j, we place a ‘1’ at the i^{th} row, j^{th} column position. Otherwise we place a ‘0’ at the appropriate position in the matrix.

Example 15

Determine the adjacency matrix for the digraph shown below,



$$
\begin{array}{c} \\ \\ \end{array}
\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}
$$

$$
A = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}
$$

Solution

- The digraph has 4 vertices and so the adjacency matrix will have dimension 4×4
- There is an edge from vertex 1 to vertex 2, so the entry in Row1/Column1 is a ‘1’
- There is an edge from vertex 2 to vertex 3, so the entry in Row2/Column3 is a ‘1’
- There is an edge from vertex 3 to vertex 2, so the entry in Row3/Column2 is a ‘1’
- There is an edge from vertex 3 to vertex 4, so the entry in Row3/Column4 is a ‘1’
- There is an edge from vertex 4 to vertex 1, so the entry in Row4/Column1 is a ‘1’
- All other entries in the adjacency matrix will be zero

Outdegree and Indegree

- In general, the number of 1’s in **row** i of A correspond to the number of edges leaving vertex i, i.e. the **outdegree** of vertex i.
- The number of 1’s in **column** j correspond to the number of edges terminating at vertex j, i.e. the **indegree** of vertex j.

For the diagraph above we can construct the following table:

Vertex	Outdegree	Indegree
1	1	1
2	1	2
3	2	1
4	1	1

Eulerian Digraphs

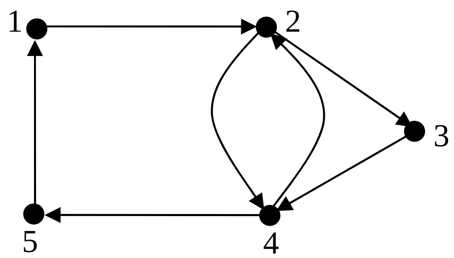
A diagraph is Eulerian if and only if the outdegree of each vertex equals its indegree. This diagraph is not Eulerian as, for example the outdegree of vertex 2 is 1 while its indegree is 2.

Notes:

- The total number of 1’s in the adjacency matrix equals the number of edges in the graph.
- In general, the adjacency matrix is not symmetric for a digraph graph.

Example 16

Determine the adjacency matrix for the digraph shown below,



$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Solution

- The digraph has 5 vertices and so the adjacency matrix will have dimension 5×5.
- There is an edge from vertex 1 to vertex 2, so the entry in Row1/Column1 is a '1'
- There is an edge from vertex 2 to vertex 3, so the entry in Row2/Column3 is a '1'
  There is an edge from vertex 2 to vertex 4, , so the entry in Row2/Column4 is a '1'
- There is an edge from vertex 3 to vertex 4, , so the entry in Row3/Column4 is a '1'
- There is an edge from vertex 4 to vertex 2, , so the entry in Row4/Column2 is a '1'
- There is an edge from vertex 4 to vertex 5, so the entry in Row4/Column5 is a '1'
- There is an edge from vertex 5 to vertex 1, so the entry in Row5/Column1 is a '1'
- All other entries in the adjacency matrix will be zero.

For the diagraph above we can construct the following table:

Vertex	**Outdegree**	**Indegree**
1	1	1
2	2	2
3	1	1
4	2	2
5	1	1

Note: This diagraph is Eulerian as the outdegree of each vertex equals its indegree.

Hamiltonian Digraphs

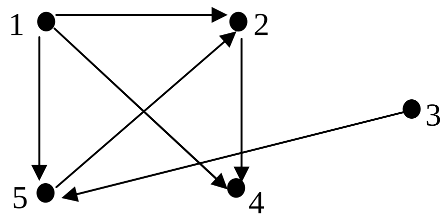
A digraph *D* is Hamiltonian if and only if there is a cycle that visits every vertex in the digraph exactly once. (Note: a cycle ends where is started).

7. Adjacency Matrices & Paths

In the previous examples the entry at position (i, j) in the adjacency matrix A, corresponds to the number of paths of length 1 between vertex v_i and vertex v_j. It is also possible to construct matrices that provide information on paths of length other than 1 between vertices. For example, to calculate the matrix for paths of length 2 we must calculate $A^2 = A \times A$.

In general, $A^k = \left(a_{ij}^{(k)}\right)$ where $a_{ij}^{(k)}$ is the number of paths of length k from i to j. Hence, the entry at position (i, j) of the matrix A^k indicates the number of paths of length k between vertex v_i and vertex v_j.

Example 17: Let G be a directed graph with 5 vertices as shown:



If a path of length 1 exists between two vertices (i.e. vertices are adjacent) then there is a 1 in the corresponding position in the adjacency matrix, A. Here, for example, inspection of A below reveals the following paths of length 1:

- from vertex 1 to vertices 2, 4 and 5
- from vertex 2 to vertex 4
- from vertex 3 to vertex 5
- from vertex 5 to vertex 2.

Note that there are no paths of length 1 from vertex 4 to any of the other vertices

Combining the above results we construct the adjacency matrix, A, for the digraph G:

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

To calculate paths of length 2 the adjacency matrix, A, is multiplied by itself to get A^2 giving a matrix representation of paths of length 2.

In this case we obtain

$$A^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

For example the matrix A^2 shows that there are only four paths of length 2 in the digraph, i.e.

from vertex 1 to vertex 2,
from vertex 1 to vertex 4,
from vertex 3 to vertex 2
from vertex 5 to vertex 4.

In general, the matrix of path length n is generated by multiplying the matrix of path length $n - 1$ by the matrix of path length 1, i.e. the adjacency matrix, A.

<u>Definition</u>: A digraph is **strongly connected** if there is a path from every vertex to every other vertex.

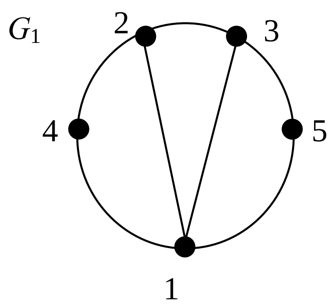
<u>8. Isomorphisms between Graphs</u>

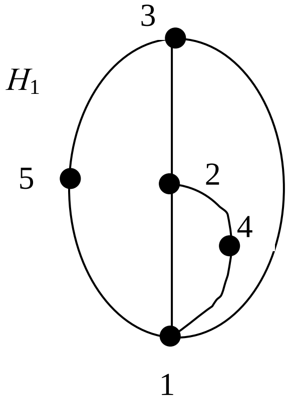
Graphs G and H are said to be **isomorphic** (essentially the same graph) if there is a one-one and onto map,

$$\phi: V(G) \to V(H) \text{ such that edge } AB \in E(G) \Leftrightarrow \text{ edge } \phi(A)\,\phi(B) \in E(H).$$

In other words there is a one-one correspondence between the vertices of G and the vertices of H with the property that the number of edges joining any two vertices of G is equal to the number of edges joining the corresponding vertices of H.

<u>Example 18</u>: The graphs G_1 and H_1 below are isomorphic.





In graph G_1: vertex 1 has degree 4 and is joined to vertices 2, 3, 4 and 5.
In graph G_1: vertex 2 has degree 3 and is joined to vertices 1, 3, and 4.
In graph G_1: vertex 3 has degree 3 and is joined to vertices 1, 2, and 5.
In graph G_1: vertex 4 has degree 2 and is joined to vertices 1 and 2.
In graph G_1: vertex 5 has degree 2 and is joined to vertices 1 and 3.

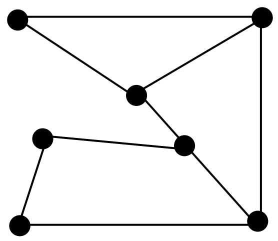
Easily checked that this is the same for graph H_1 and so the graphs are isomorphic.

Hence, the **adjacency list** is the same for both graphs

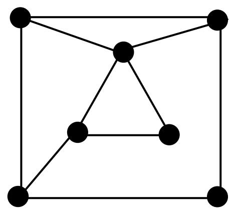
Vertex	**Adjacent vertices**
1	2, 3, 4, 5
2	1, 3, 4
3	1, 2, 5
4	1, 2
5	1, 3

Example 19: The graphs G_2 and H_2 below are not isomorphic as they have different degree sequences.

G_2



H_2



Both graphs have the same number of vertices, i.e. 7. However, Graph G_2 has degree sequence (2, 2, 2, 3, 3, 3,3) while Graph G_2 has degree sequence (2, 2, 3, 3, 3, 3, 4). Alternatively you could show that the two graphs have different adjacency lists.

9. Vertex (Graph) Colouring

The most well-known graph colouring problem is the Four Colour Problem which was first proposed in 1852 when Francis Guthrie noticed that four colours were sufficient to colour a map of the counties of England so that no two counties with a border in common had the same colour. Guthrie conjectured that any map, no matter how complicated, could be coloured using at most four colours so that adjacent regions (regions sharing a common boundary segment, not just a point) are not the same colour. Despite many attempts at a proof it took until 1976 when two American scientists, Appel and Haken, using graph theory produced a computer-based proof to what had become known as the Four Colour Theorem.

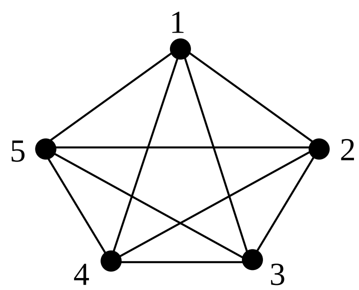
In graph theory terms vertex (graph) colouring problems require the assignment of colours (usually represented by integers) to the vertices of the graph so that no two adjacent vertices are assigned the same colour (integer).

Definition

A k-colouring of a graph is a colouring in which only k colours (numbers) are used. The **chromatic number** for a graph is the **minimum** number of colours (numbers) required to produce a vertex colouring of the graph. The chromatic number of a graph G is denoted by $\chi(G)$.

Example 20

A graph with no edges has chromatic number 1 while the complete graph K_n has chromatic number n. In the figures below we assign a '1' to the graph with no edges on the left and say that it is 1-colourable while we assign the numbers 1, 2, 3, 4, 5 to the complete graph K_5 on the right and say that it is 5-colourable.



Identifying the chromatic number in the two cases shown above is straightforward. In general, however determining the exact chromatic number of a graph is a hard problem and no efficient method exists. The only approach that would identify the chromatic number of a graph G with absolute certainty would involve investigating all possible colourings. Clearly as graphs become larger this method becomes impractical, even using the most powerful computers that are available. The best that can be done is to determine lower and upper bounds on the chromatic number and techniques such as looking for the largest complete subgraph in G (for a lower bound) and the Greedy algorithm (for an upper bound) enables us to do so. The Greedy algorithm however is very inefficient but is adequate for 'small' graphs with the aid of a computer.

TUTORIAL

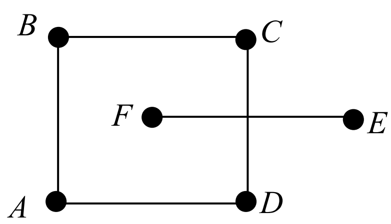
1. Sketch the following graphs:

(i). 4-regular, **(ii).** 5-regular, **(iii).** K_4,

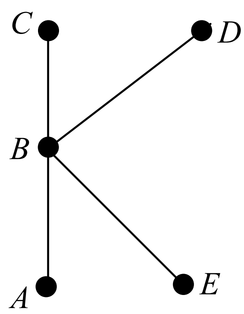
(iv). C_6, **(v).** $K_{2,3}$ **(vi).** $K_{4,4}$

2. (i). Which of the following graphs are connected?

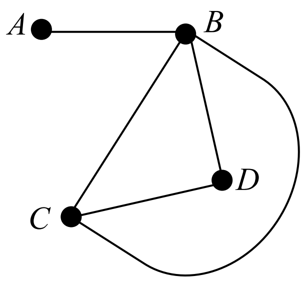
(a).



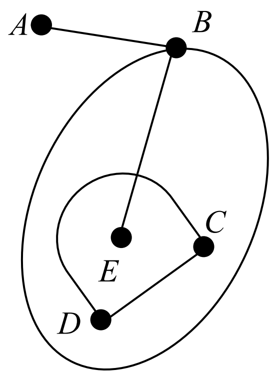
(b).



(c).



(d).



(ii). If a graph is not connected state what its connected components are.

(iii). Which are simple graphs and which are multigraphs?

3. Sketch the undirected graph G defined below and construct the adjacency matrix.

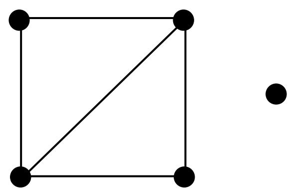
G = { V, E } = { { 1, 2, 3,4, 5 }, {(1, 2), (1, 3), (1, 5), (1, 5), (2, 1), (2, 3), (2, 3), (3, 1), (3, 2), (3, 2), (3, 4), (3, 5), (4, 3), (4, 5), (5, 1), (5, 1), (5, 3), (5, 4)} }.

4. Consider the adjacency matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(i). Sketch the associated undirected graph *G*.
(ii). Write down the degree sequence for *G*.
(iii). Show that the Handshaking Lemma holds for *G*.
(iv). Is *G* Eulerian? Justify your answer and give an Euler circuit if appropriate.
(v). Is *G* Hamiltonian? Justify your answer and give a Hamiltonian circuit if appropriate.
(vi). Removal of an edge from *G* results in a bipartite graph. Identify which edge should be removed and sketch the resulting graph.
(vii). How many edges need to be added to *G* to obtain a complete graph? Identify which edges need to be added and sketch the resulting graph.

5. Given a graph, *G*, its complementary graph $\overline{G}$, obtained from *G* by replacing edges with non-edges and non-edges by edges. If *G* is given by:

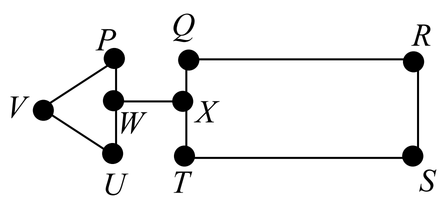


Sketch its complementary graph, $\overline{G}$.

6. A graph, *G*, is *k*-regular if all vertices have degree *k*. Calculate the degree sum for a *k*-regular graph with *n* vertices and the number of edges in *G*.

7. In a simple graph, with at least two vertices, there are at least two vertices of the same degree. This result is not true for multigraphs. Sketch a three vertex multigraph with all vertices of different degree.

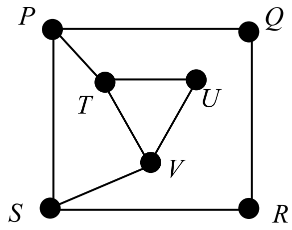
8. Consider the graph, *G* below. Explain why *G* does not have a Hamiltonian circuit.



9. Define the term Hamiltonian cycle (circuit) and sketch a Hamiltonian graph.

10. Define the term Euler circuit and sketch an Eulerian graph.

11. Consider the graph below,

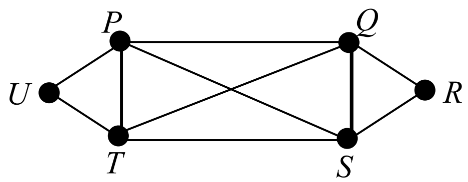


(i). Is the graph Eulerian? If so give an Euler circuit of *G*.

(ii). Is the graph Hamiltonian? If so give a Hamiltonian circuit of *G*.

12. Sketch a simple graph *G* whose vertices all have even degree but *G* is not Eulerian.

13. Consider the graph *G* below,

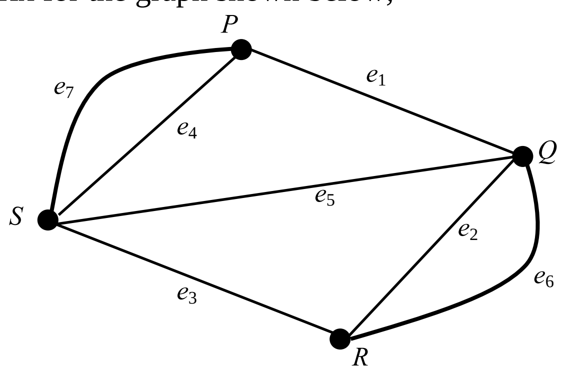


(i). Is *G* Eulerian? If so give an Euler circuit of *G*.

(ii). Is *G* Hamiltonian? If so give a Hamiltionian circuit of *G*.

14. Determine whether the complete graphs K_{77} and K_{32} are Eulerian.

15. Determine the adjacency matrix for the graph shown below,



16. The adjacency matrix for a graph, G is given by

$$A = \begin{bmatrix} 2 & 1 & 1 & 3 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 3 & 1 & 1 & 0 \end{bmatrix}.$$

Without drawing G, and using only the matrix A, answer the following:

(i) How many edges does G have?

(ii) How many paths of length 2 join vertices A and D.

17. How many edges does a tree, T, with 5000 vertices have?

18. Determine which bipartite graphs, $K_{m,n}$ are trees.

19. Determine the conditions on r and s that will guarantee that the complete bipartite graph, $K_{r,s}$ will have an Euler circuit.

20. Explaining your answer state whether a graph on 7 vertices can have each vertex of degree 5.

21. Consider a graph G on 12 vertices where each vertex has degree 7. How many edges does G have? Explain your answer.

22. (i) Sketch the digraph D = { {1, 2, 3, 4}, (1,2), (1,4), (2,3), (2,4), (3,2), (3,4), (4,1) }.

(ii) Determine the adjacency matrix for D.

(iii) If the adjacency matrix A satisfies, $A^2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ calculate A^3 and explain the meaning of the entry at position (1, 2) in A^3.

23. Consider the following adjacency matrix, A, for a directed graph, G

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Without drawing G, and using only the matrix A answer the following:

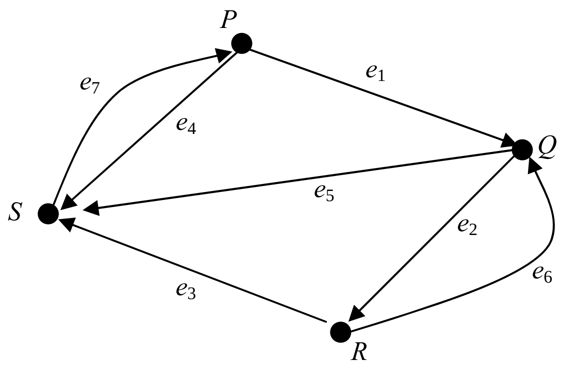
(i) Calculate the indegree and outdegree of each vertex.

(ii) Determine whether G is Eulerian. Explain your answer.

(iii) How many edges does G have? Explain your answer.

24. State the Handshaking Lemma for directed graphs, explaining your answer.

25. Determine the adjacency matrix for the digraph below.

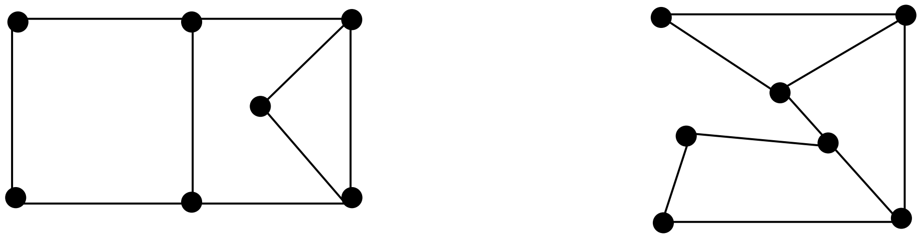


26. Consider the following adjacency matrix $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$.

(i). Sketch the associated digraph.

(ii). Determine whether the digraph is Eulerian and state an Euler circuit if one exists.

27. Determine whether the two graphs below are isomorphic.



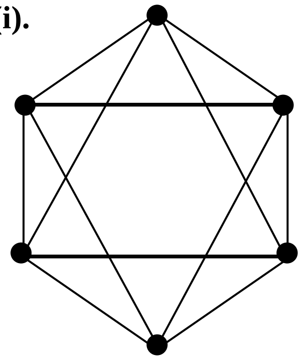
28.(i). In a cycle graph C_n state how the number of vertices is related to the number of edges.

(ii). Sketch the cycle graphs C_5 and C_6.

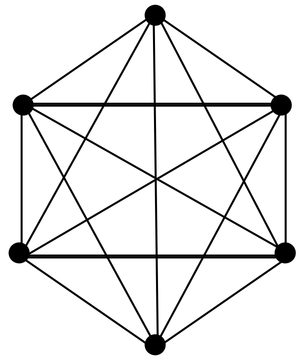
(iii). What is the chromatic number of a cycle graph, C_n?

Solutions

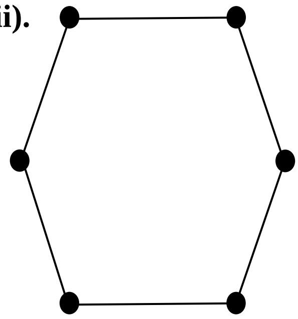
1.(i).



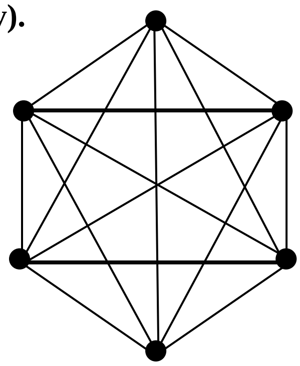
(ii).



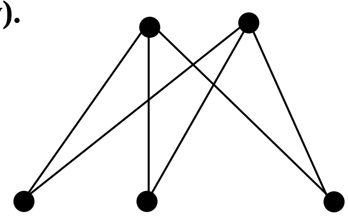
(iii).



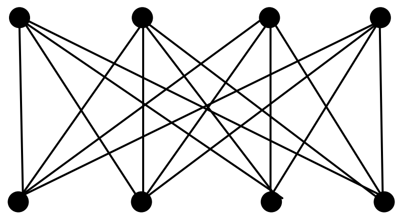
(iv).



(v).



(vi).

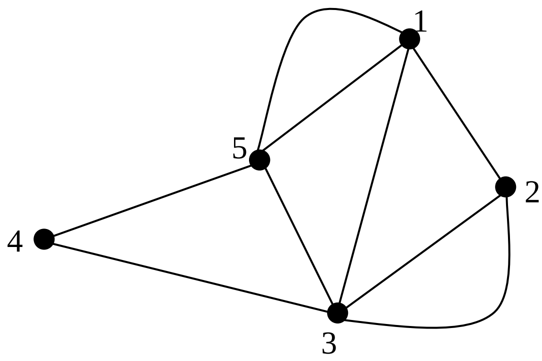


2. (i). Graphs (b) and (c) are connected as there is a path between any two of their vertices.

(ii). Graph (a) is disconnected and its disconnected components are {*ABCD*} and {*EF*}.
Graph (d) is disconnected and its disconnected components are {*ABE*} and {*CD*}

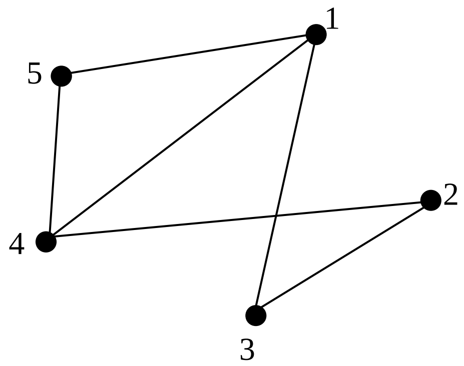
(iii). Graphs (a) and (b) are simple graphs.
Graph (c) is a multigraph with multiple edges (*BC*) and (*BC*).
Graph (d) is a multigraph with multiple edges (*CD*) and (*CD*) and a self-loop (*BB*).

3.



$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 & 0 \\ 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 \end{bmatrix}$$

4. (i).

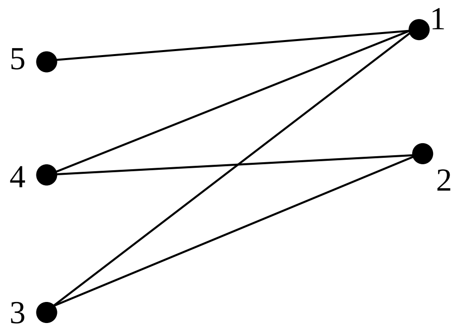


(ii). degree sequence, (2, 2, 2, 3, 3)

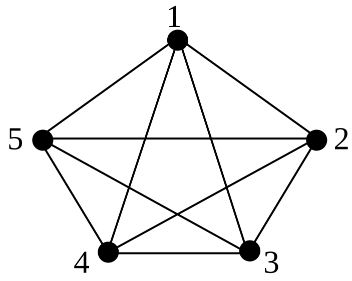
(iii). By the Handshaking Lemma $\sum_{j=1}^{n} \deg(v_j) = 2|E(G)|$ where $|E(G)|$ is the number of edges in G. We therefore have $\sum_{j=1}^{5} \deg(v_j) = 2+2+2+3+3 = 12$ and $2|E(G)| = 2\times 6 = 12$. Hence, the Handshaking Lemma holds for G.

(iv). G is not Eulerian as not all the vertices have even degree.

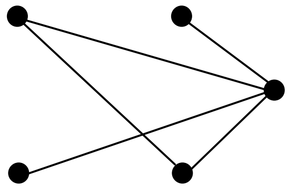
(v). Removal of edge (4, 5) results in the bipartite graph below.



(vi). Adding the four edges (1, 2), (2, 5). (3, 4), (3, 5) results in the complete graph K_5.

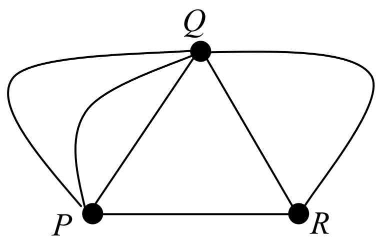


5. The complementary graph, $\overline{G}$ is



6. The regular graph G has n vertices all of degree k and so the sum of all the degrees is nk. By the Handshaking Lemma $\sum_{j=1}^{n} \deg(v_j) = 2|E(G)|$ where $|E(G)|$ is the number of edges in G. We therefore have $kn = 2|E(G)| \Rightarrow |E(G)| = \frac{kn}{2}$.

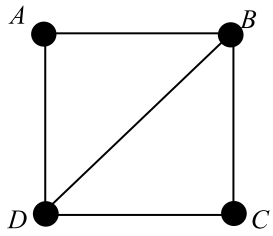
7. In the graph below; deg(P) = 4, deg(Q) = 5, deg(R) = 3



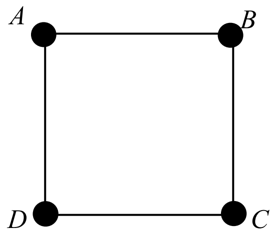
8. A Hamiltonian circuit visits each vertex exactly once and returns to the starting vertex. Note that G consists of two subgraphs *PUV* and *QRST* connected by a bridge *WX*. If we start on the left-hand-side (*PUV*) we must cross the bridge (*WX*) in order to visit every vertex on the right-hand-side but to get back to our starting vertex we must cross the bridge again thereby visiting the vertices *X* and *W* for a second time. Therefore G does not have a Hamiltonian circuit.

Note: No graph with a bridge has a Hamiltonian circuit.

9. A Hamiltonian circuit visits each vertex exactly once and returns to the starting vertex. The graph below is Hamiltonian and a Hamiltonian circuit is: *ABCDA*. Note that we do not need to use all edges.



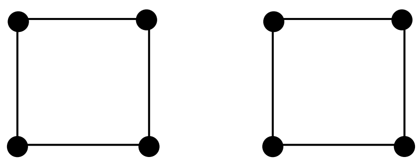
10. An Euler circuit is a path through a connected graph which starts and ends at the same vertex and travels along every edge of the graph exactly once. The graph below is Eulerian and an Euler circuit is: *ABCDA*.



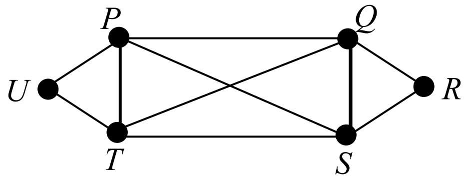
11. (i). The graph is not Eulerian as it contains vertices of odd degree, i.e. vertices *P*, *S* and *T* all have degree 3.

(ii). The graph is Hamiltonian and a Hamiltonian circuit is, *PTUVSRQP*.

12. For example, the graph below has every vertex of *n* degree 2 but it is not Eulerian as it is disconnected.



13. (i). Eulerian: Yes as all vertices have even degree. Euler circuit: *PSRQSTUPTQP*.
(ii). Hamiltonian: Yes. Hamiltonian circuit: *PQRSTUP*.



14. The graph K_{77} is 76-regular and all vertices therefore have even degree so that, by Euler's theorem, K_{77} is Eulerian.

The graph K_{32} is 31-regular and all vertices therefore have odd degree so that K_{32} is not Eulerian.

15 (i). $A = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix}$ **(ii)** $M = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$

16 (i). Number of edges in $G = |E(G)| = \frac{1}{2}\sum_{X \in V(G)} \deg(X) = \frac{1}{2} \times 20 = 10$.

(ii) For the number of paths of length 2 joining vertices A and D we must calculate A^2 .

$$A^2 = \begin{bmatrix} 2 & 1 & 1 & 3 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 3 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 3 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 3 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 15 & 7 & 7 & 8 \\ 7 & 6 & 2 & 5 \\ 7 & 2 & 6 & 5 \\ 8 & 5 & 5 & 11 \end{bmatrix}$$

There are 8 paths of length 2 joining vertices A and D

17. Note that T is a tree so that, by definition, T is cycle-free and has n – 1 edges.
As $|V|$ = 5000 then $|E|$ = 5000 – 1 = 4999

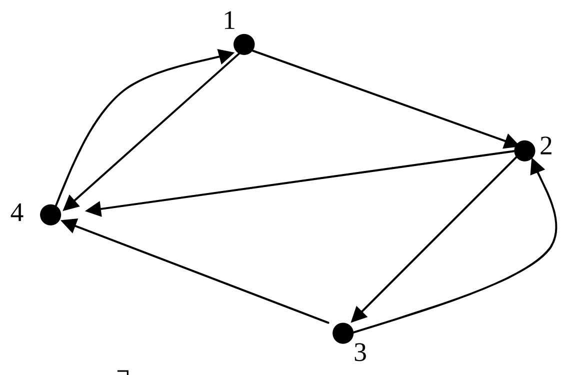
18. If m = 1 and/or n = 1 then $K_{m,n}$ is a tree.

19. If r and s are both even the complete bipartite graph, $K_{r,s}$ will have an Euler circuit as each vertex will have even degree.

20. By the Handshaking Lemma this is not possible as the sum of the degrees of the vertices, i.e. $7 \times 5 = 35$, which is odd.

21. By the Handshaking Lemma the degree sum is twice the number of edges. Hence, since degree sum = $12 \times 7 = 84$ we have that $2E$ = 84 and so the number of edges E = 42.

22. (i).



(ii). $A=\begin{bmatrix}0&1&0&1\\0&0&1&1\\0&1&0&1\\1&0&0&0\end{bmatrix}$ **(iii).** $A^3=\begin{bmatrix}1&2&0&2\\1&1&1&2\\1&2&0&2\\1&0&1&1\end{bmatrix}$

The entry at position (1, 2) in A^3 indicates that there are exactly two paths of length 3 from vertex 1 to vertex 2, i.e. 1412 and 1232.

23. (i). Label the rows and columns of the matrix, *P*, *Q*, *R*, *S*, *T* from top to bottom and *P*, *Q*, *R*, *S*, *T* from left to right. The sum of the entries in row *j* corresponds to the outdegree of vertex *j*.
The sum of the entries in column *j* corresponds to the indegree of vertex *j*.

	P	*Q*	*R*	*S*	*T*
Outdegree	1	2	3	2	2
Indegree	2	2	2	2	2

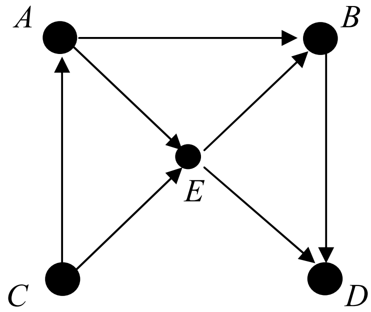
(ii). No, *G* is not Eulerian as the indegree does not equal the outdegree for each vertex.

(iii). The graph *G* has 10 edges as each 1 in the adjacency matrix corresponds to an edge.

24. For directed graphs the Handshaking Lemma states that the sum of the indegrees is equal to the sum of the outdegrees and the combined total is equal to number of edges. This is because every edge counts exactly once to the outdegree total and exactly once to the indegree total.

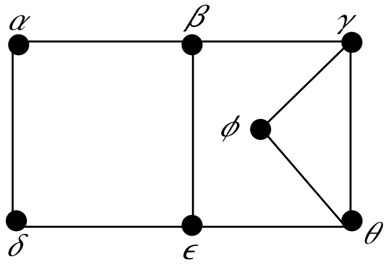
25.(i). $A=\begin{bmatrix}0&1&0&1\\0&0&1&1\\0&1&0&1\\1&0&0&0\end{bmatrix}$

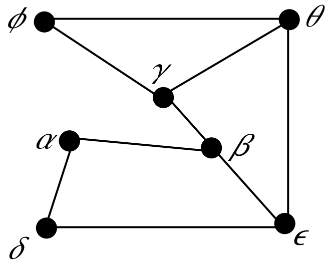
26. (i).



(ii) No, G is not Eulerian as the indegree does not equal the outdegree for each vertex. We can determine this from the adjacency matrix.

27. The graphs are isomorphic under the correspondence shown.



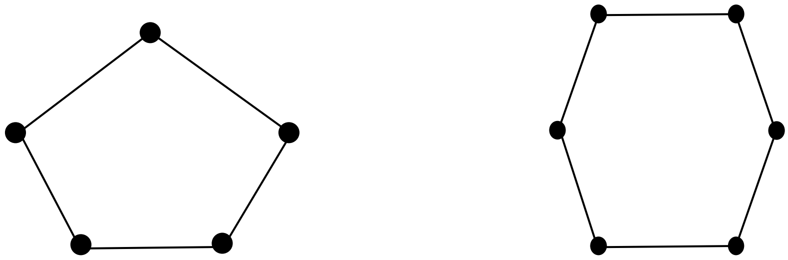


The adjacency list is the same for both graphs

α	β, δ
β	$\alpha, \gamma, \varepsilon$
γ	β, φ, θ
φ	γ, θ
θ	$\varepsilon, \gamma, \varphi$
ε	δ, θ, β
δ	α, ε

28(i). The number of vertices in C_n equals the number of edges, and every vertex has degree 2.

(ii). The cycle graphs C_5 and C_6 are shown below.



(iii). The chromatic number of a cycle graph, C_n, is 2 if n is even and 3 if n is odd.