School of Computing, Engineering & Built Environment

Mathematics Summer School

Level 2 Entry – Engineering
&
Level 3 Entry – Computing

Basic Algebra

Contents Page

1. Operations and Expressions

You will already be familiar with various forms of mathematical expressions. It is important that you are confident in handling and manipulating such expressions, so this section should refresh your memory regarding some basic algebraic concepts and techniques.

(a). The Basic Arithmetic Operations

In mathematical expressions, numbers and / or variables are combined using the arithmetic operations of addition, subtraction, multiplication, division and exponentiation (e.g. squaring, cubing, etc.), along with brackets to group or separate terms. The order in which we carry out a calculation is very important. For example, if we were asked to evaluate,

$$9 + 4 \times 3$$

we could interpret this calculation in two ways:

(i). carry out the addition first followed by the multiplication, i.e.

$$9 + 4 \times 3 = 13 \times 3 = 39$$

(ii). carry out the multiplication first followed by the addition, i.e.

$$9 + 4 \times 3 = 9 + 12 = 21 .$$

We have obtained two different answers for the same expression, which is unrealistic, and would therefore like to know which of our approaches is correct. It turns out that there is a rule that defines the order in which mathematical operations are performed.

The acronym BODMAS describes the following rule for the evaluation of mathematical expressions :

Brackets	first priority
Orders (i.e. powers)	second priority
Division, **M**ultiplication	third priority
Addition, **S**ubtraction	fourth priority.

In the above example the second method, giving the answer 21, is the correct approach using the BODMAS rule as multiplication has priority over addition.

The BODMAS rule states that calculations inside brackets are performed first, followed by calculations involving powers (indices) or roots. For example,

$$(2 + 4)^3 = 6^3 = 216.$$

Multiplication and division are ranked next and have equal priority. We evaluate expressions involving these operators from **left to right**. For example,

$$\begin{aligned} &2 \times 9 \div 3 \times 7 \\ &= 18 \div 3 \times 7 \\ &= 6 \times 7 = 42. \end{aligned}$$

Finally addition and subtraction have equal priority and are performed last. We evaluate expressions involving these operators from **left to right**. For example,

$$\begin{aligned} &2 + 5 - 9 + 6 \\ &= 7 - 9 + 6 \\ &= -2 + 6 = 4. \end{aligned}$$

Here is one more example of the BODMAS rule:
Evaluate $4 + 6 \div 2 \times (5 - 1)^2 - 9$.

$$\begin{aligned} &(B) && 4 + 6 \div 2 \times 4^2 - 9 \\ &(O) && = 4 + 6 \div 2 \times 16 - 9 \\ &(D) && = 4 + 3 \times 16 - 9 \\ &(M) && = 4 + 48 - 9 \\ &(A) && = 52 - 9 \\ &(S) && = 43. \end{aligned}$$

To remove any ambiguity we could write the expression using brackets and evaluate the brackets from the inside-out, i.e.

$$\begin{aligned} &(4 + [(6 \div 2) \times (5 - 1)^2]) - 9 \\ &= (4 + [3 \times 4^2]) - 9 \\ &= (4 + [3 \times 16]) - 9 \\ &= (4 + 48) - 9 \\ &= 52 - 9 = 43. \end{aligned}$$

The correct use of brackets is extremely important in mathematics.

(b). Indices (Powers)

The simple use of powers or indices to represent repeated multiplication should be familiar, for example

$$
\begin{aligned}
a^2 &= a \times a \\
a^3 &= a \times a \times a \\
a^n &= a \times a \times a \;.\;\;.\;\;. \times a \qquad [n \text{ terms}],
\end{aligned}
$$

but you must also be clear on the wider interpretation of indices. The lines below collate the important aspects of indices:

- $a^n \, a^m = a^{n+m}$
- $\dfrac{a^n}{a^m} = a^{n-m}$
- $(a^n)^m = a^{mn}$
- $(ab)^m = a^m \, b^m$
- $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$
- $a^0 = 1$
- $a^{-n} = \dfrac{1}{a^n}$ and $\dfrac{1}{a^{-n}} = a^n$
- $a^{1/2} = \sqrt{a}$ and $a^{1/m} = \sqrt[m]{a}$.

These results will be needed in both the expansion and simplification of algebraic expressions.

Examples

E1. **(i).** Expand $(3x^2 + 7y^2)(5x^2 - 6y^2)$.

$$\begin{aligned}(3x^2 + 7y^2)(5x^2 - 6y^2) &= 3x^2(5x^2 - 6y^2) + 7y^2(5x^2 - 6y^2) \\ &= 15x^4 - 18x^2y^2 + 35y^2x^2 - 42y^4 \\ &= 15x^4 + 17x^2y^2 - 42y^4\end{aligned}$$

(ii). Expand $(x + 4)^3$.

$$\begin{aligned}(x + 4)^3 &= (x + 4)(x + 4)^2 \\ &= (x + 4)(x^2 + 8x + 16) \\ &= x(x^2 + 8x + 16) + 4(x^2 + 8x + 16) \\ &= x^3 + 8x^2 + 16x + 4x^2 + 32x + 64 \\ &= x^3 + 12x^2 + 48x + 64\end{aligned}$$

(iii). Simplify $\dfrac{15\,x\,(y z^2)^3}{3\,x^4 y^2 z^8}$.

$$\begin{aligned}\frac{15\,x\,(y z^2)^3}{3\,x^4 y^2 z^8} &= \frac{15\,x y^3 z^6}{3\,x^4 y^2 z^8} \\ &= 5\,x^{-3} y^1 z^{-2} \\ &= \frac{5\,y}{x^3 z^2}\end{aligned}$$

See the Tutorial Exercises for further practice.

2. Common Mistakes

Here are just a few common mistakes to avoid when manipulating algebraic expressions:

✗		✓
• $(2x)^3 = 2x^3$	:	$(2x)^3 = 2^3 x^3 = 8x^3$
• $(x+y)^2 = x^2 + y^2$	:	$(x+y)^2 = x^2 + 2xy + y^2$
• $\frac{a+b}{c+d} = \frac{a}{c} + \frac{b}{d}$	:	$\frac{a+b}{c+d} = \frac{a}{c+d} + \frac{b}{c+d}$
• $a(bc) = ab\,ac$	:	$a(bc) = abc$
• $a^{m+n} = a^m + a^n$	:	$a^{m+n} = a^m a^n$
• $\frac{1}{2x^3} = 2x^{-3}$	:	$\frac{1}{2x^3} = \frac{1}{2}x^{-3}$

3. Division of Algebraic Expressions

Providing the denominator (i.e. the bottom bit) of a division contains no +'s or –'s, the division itself is usually straightforward to carry out.

Examples

E2. (i).

$$\frac{4a^3 - 10a^2 + 6a}{2a} = \frac{4a^3}{2a} - \frac{10a^2}{2a} + \frac{6a}{2a}$$

$$= 2a^2 - 5a + 3$$

(ii).

$$\frac{8x^2y^3 - 24x^3y^2}{4xy^2} = \frac{8x^2y^3}{4xy^2} - \frac{24x^3y^2}{4xy^2}$$

$$= 2xy - 6x^2$$

$$= 2x(y - 3x)$$

When the denominator does contain +'s and / or –'s, we may require polynomial division.

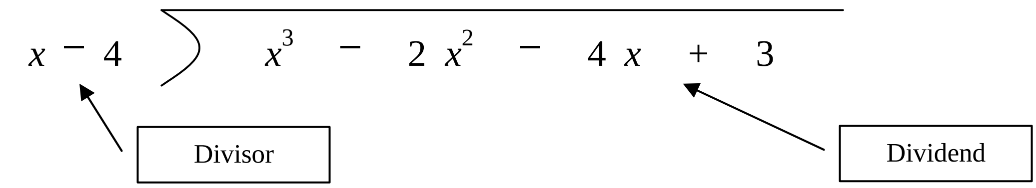
Note: $2x+3$, $x^2 - 4x + 5$, $x^3 + 3x^2 - 2x + 6$ are examples of polynomials.

Polynomial Division Example

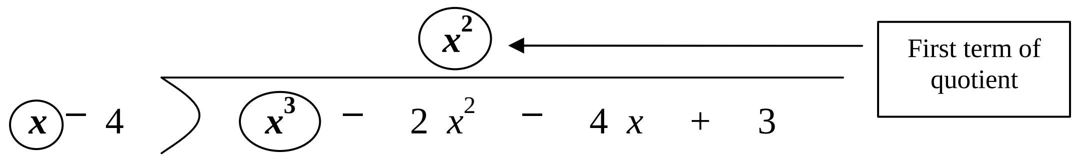
Suppose we want to carry out the following polynomial division:

$$\frac{x^3 - 2x^2 - 4x + 3}{x - 4}.$$

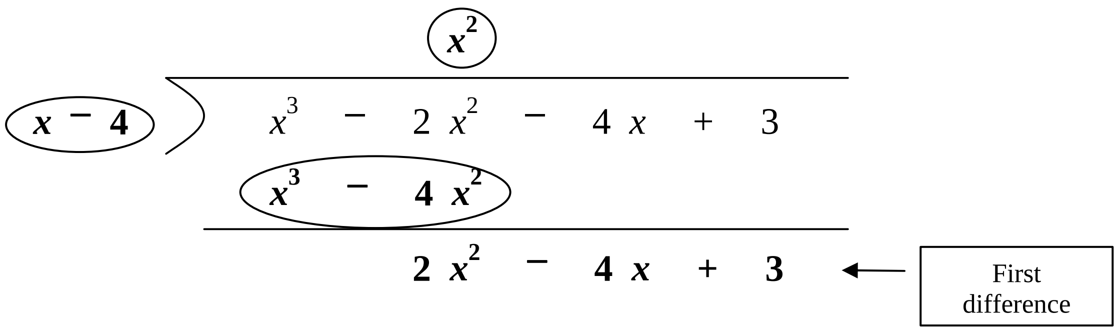
First we set it out like a long-division:



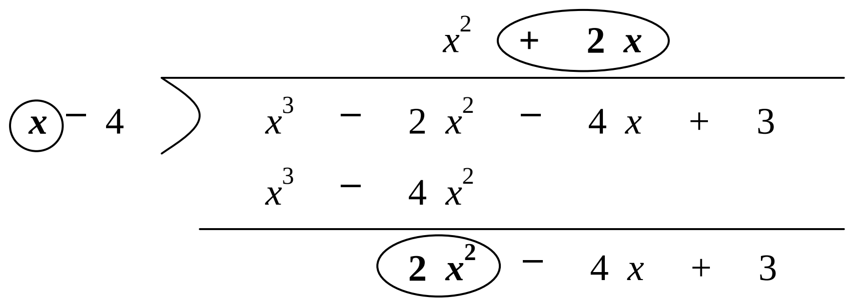
Next, we divide the first term of the dividend by the first term of the divisor and place the result above the line, keeping corresponding powers of x vertically aligned:



Multiply the divisor by the term of the quotient just found. Align the result below the dividend and subtract:



Repeat the process from the first difference to give the next term of the quotient. Divide first terms:



Multiply divisor and subtract:

$$
\begin{array}{r|l}
 & x^2 + 2x \\
\hline
x - 4 & x^3 - 2x^2 - 4x + 3 \\
 & x^3 - 4x^2 \\
\cline{2-2}
 & \qquad 2x^2 - 4x + 3 \\
 & \qquad 2x^2 - 8x \\
\cline{2-2}
 & \qquad\qquad 4x + 3 \quad \leftarrow \text{Second difference}
\end{array}
$$

Repeat to complete quotient and determine the remainder:

$$
\begin{array}{r|l}
 & x^2 + 2x + 4 \\
\hline
x - 4 & x^3 - 2x^2 - 4x + 3 \\
 & x^3 - 4x^2 \\
\cline{2-2}
 & \qquad 2x^2 - 4x + 3 \\
 & \qquad 2x^2 - 8x \\
\cline{2-2}
 & \qquad\qquad 4x + 3
\end{array}
$$

$$
\begin{array}{r|l}
 & x^2 + 2x + 4 \\
\hline
x - 4 & x^3 - 2x^2 - 4x + 3 \\
 & x^3 - 4x^2 \\
\cline{2-2}
 & \qquad 2x^2 - 4x + 3 \\
 & \qquad 2x^2 - 8x \\
\cline{2-2}
 & \qquad\qquad 4x + 3 \\
 & \qquad\qquad 4x - 16 \\
\cline{2-2}
 & \qquad\qquad\qquad 19 \quad \leftarrow \text{Remainder}
\end{array}
$$

Removing the annotation we have

$$
\begin{array}{r|rrrr}
 & & x^2 & +\ 2x & +\ 4 \\
\hline
x-4 & x^3 & -\ 2x^2 & -\ 4x & +\ 3 \\
 & x^3 & -\ 4x^2 & & \\
\cline{2-5}
 & & 2x^2 & -\ 4x & +\ 3 \\
 & & 2x^2 & -\ 8x & \\
\cline{3-5}
 & & & 4x & +\ 3 \\
 & & & 4x & -\ 16 \\
\cline{4-5}
 & & & & 19
\end{array}
$$

.

In general,

$$\frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}} ,$$

so for the above example we can write

$$\frac{x^3 - 2x^2 - 4x + 3}{x - 4} = x^2 + 2x + 4 + \frac{19}{x - 4} .$$

Important note:

Care must be taken when any of the polynomials have “missing terms”. These should be included with zero coefficients to ensure correct alignment and to minimise mistakes. For example, if we have

$$x^3 + 2x^2 - 1 ,$$

then this should be written as

$$x^3 + 2x^2 + 0x - 1 .$$

4. Exponential Functions and Logarithms

An exponent is another name for a power or index. Exponents can be used to create exponential functions:

$$y = a^x \quad , \quad (a > 0 : \; a \neq 0, 1).$$

In this context, a is called the **base** of the exponential function. Commonly used bases are 10 and the exponential constant $e \approx 2.71828$:

$$y = 10^x \; ;$$

$$y = e^x \; .$$

The function with the exponential constant e as its base is so important in mathematics, science and engineering that it is referred to as **the** exponential function, all others being subordinate.

Related to exponential functions are logarithms. If we express a number N as a power of a, i.e.

$$N = a^x \; ,$$

then the power is defined to be the (base a) logarithm of N :

$$x = \log_a N \qquad (\log \text{ of } N \text{ to the base } a)$$

Note that because of the importance of the exponential constant, logarithms to base e are given the special name of **natural logarithms** and denoted by $\ln(\;)$.

Examples

E3. (i). $100 = 10^2 \quad \rightarrow \quad \log_{10}(100) = 2$

(ii). $1000 = 10^3 \quad \rightarrow \quad \log_{10}(1000) = 3$

(iii). $64 = 2^6 \quad \rightarrow \quad \log_2(64) = 6$

The definition can be extended to fractional powers with logarithms to base 10 and base e obtainable from calculators:

E4. (i). $\log_{10}(500) = 2.698970$ (to 6 decimal places)

(ii). $\ln(42) = 3.7376696$ (to 7 decimal places)

Because logs are by definition indices, we can use the rules for combining indices to determine the so-called laws of logarithms:

- $\log_a(RS) = \log_a R + \log_a S$ [L1]
- $\log_a\left(\dfrac{R}{S}\right) = \log_a R - \log_a S$ [L2]
- $\log_a(R^n) = n\log_a R$. [L3]

We can use these to expand or contract expressions involving logarithms:

Examples

E5. (i). Expand $\log_{10}\left(\dfrac{x^3 y^4}{z^2}\right)$.

$$\begin{aligned}\log_{10}\left(\frac{x^3 y^4}{z^2}\right) &= \log_{10}(x^3 y^4) - \log_{10}(z^2) && \text{[L2]}\\ &= \log_{10}(x^3) + \log_{10}(y^4) - \log_{10}(z^2) && \text{[L1]}\\ &= 3\log_{10}(x) + 4\log_{10}(y) - 2\log_{10}(z) && \text{[L3]}\end{aligned}$$

(ii). Write $4\ln(x) + 6\ln(y+z) - 3\ln(y)$ as a single logarithm.

$$\begin{aligned}4\ln(x) + 6\ln(y+z) - 3\ln(y) &= \ln(x^4) + \ln[(y+z)^6] - \ln(y^3) && \text{[L3]}\\ &= \ln[x^4(y+z)^6] - \ln(y^3) && \text{[L1]}\\ &= \ln\left[\frac{x^4(y+z)^6}{y^3}\right] && \text{[L2]}\end{aligned}$$

We shall return to exponentials and logs later.

5. Operations and their Inverses

Each basic arithmetic operation that we may encounter (with a few exceptions) has associated with it a corresponding inverse operation. An operation and its inverse, when applied in sequence, effectively cancel out one and other. For example, suppose we start off with x . If we now add a and then subtract a , we are back to x again. That is,

$$x + a - a = x \ .$$

We can say that the inverse of adding a is subtracting a (and also vice versa!).

Similarly, if we multiply x by a , then divide by a , ($a \neq 0$) we are again back to x :

$$\frac{a\,x}{a} = x \ .$$

So the inverse of multiplying by a is dividing by a (and vice versa).

The table below shows operations and their inverses, including the exponentials and logs from the previous section, and some others you may already be familiar with:

$x + a - a = x$	$x - a + a = x$
$\frac{a\,x}{a} = x$	$\frac{x}{a}\,a = x$
$\frac{b}{a}\,\frac{a}{b}\,x = x$	This is a special case of the entry above. Multiplying by $\frac{a}{b}$ is the same as multiplying by *a* and dividing by *b*. Hence the inverse operation is multiplying by $\frac{b}{a}$.
$\sqrt{x^2} = x$ or $(x^2)^{1/2} = x$	$\left(\sqrt{x}\right)^2 = x$ or $(x^{1/2})^2 = x$
$\sqrt[3]{x^3} = x$ or $(x^3)^{1/3} = x$	$\left(\sqrt[3]{x}\right)^3 = x$ or $(x^{1/3})^3 = x$
$\sqrt[n]{x^n} = x$ or $(x^n)^{1/n} = x$	$\left(\sqrt[n]{x}\right)^n = x$ or $(x^{1/n})^n = x$
$10^{\log_{10} x} = x$	$\log_{10}(10^x) = x$
$e^{\ln x} = x$	$\ln(e^x) = x$
$\sin^{-1}(\sin x) = x$	$\sin(\sin^{-1} x) = x$
$\cos^{-1}(\cos x) = x$	$\cos(\cos^{-1} x) = x$
$\tan^{-1}(\tan x) = x$	$\tan(\tan^{-1} x) = x$

6. Manipulating Formulae and Solving Equations

The operation / inverse operation effect described in the previous section provides the key to manipulating formulae and solving equations.

A formula is an equation that expresses a relationship between "quantities". In particular, it expresses how to determine the value of one quantity from the values of one or more other quantities. Below are some examples of formulae that you may have seen before:

- $C = \frac{5}{9}(F - 32)$
- $V = \frac{4}{3}\pi r^3$
- $d = ut + \frac{1}{2}at^2$
- $v = u + at$
- $v = iR$

In each of these formulae, the quantity on the left is called the **subject** of the formula. Often, when working with a formula, we want to change the subject to one of the other quantities. For example:

$$v = iR \quad \rightarrow \quad i = \frac{v}{R} \quad - \quad i \text{ is now the subject.}$$

This is a very simple example. However, the manipulation of formulae is an aspect of algebra that can pose difficulties for students.

Consider the formula in the above list that relates temperature in degrees Celsius to temperature in degrees Fahrenheit:

$$C = \tfrac{5}{9}(F - 32) \ .$$

It is quite easy to make F the subject of this formula. How you would tackle this would probably depend on methods brought from school. If you are thinking something like, "move terms from one side of the equation to the other to get F on its own", then please think again. Although the "method" of "moving" terms will probably get you the correct answer in this case, it is not a mathematically correct way of thinking and can cause problems in more complicated formulae.

Let us now look at the correct way to manipulate a formula or equation. This will let you see what is really happening when terms appear to "move" around an equation.

Warning: Ignore the following at your peril!

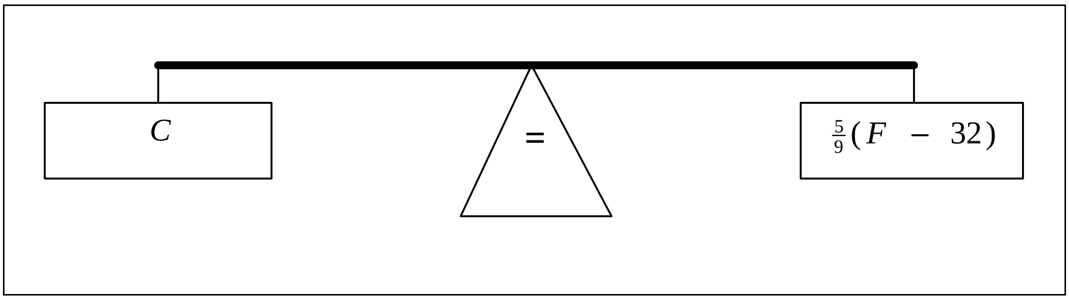
First, let us look at the formula as given and the mathematical operations within it:

$$C = \tfrac{5}{9}(F - 32) \;.$$

Given a value of F, (following BODMAS) we would determine the corresponding value of C by:

- subtracting 32
- multiplying by $\tfrac{5}{9}$.

A formula (or, in fact, any equation) is like a balanced set of scales:



When working with an equation, we must not upset the balance. This means that if we do something to one side of the equation, then we must do the exact same thing to the other side. This is the one and only rule of algebraic manipulation. [Note: We can, of course, swap the sides around, but that is just like turning the scales around; it does not affect the balance.]

To change the subject of a formula, we apply carefully chosen operations to **both sides of the equation** whose net effect is to isolate the new subject. These operations are the inverse operations of the those within the original formula. That is:

- subtract 32
- multiply by $\tfrac{5}{9}$

are reversed and inverted to give

- multiply by $\tfrac{9}{5}$
- add 32 .

These operations are now applied, in turn, to both sides of the equation.

The process in full is as follows:

$$C = \tfrac{5}{9}(F - 32)$$

Multiply both sides by $\tfrac{9}{5}$:

$$\tfrac{9}{5}C = \tfrac{9}{5}\tfrac{5}{9}(F - 32)$$

$$\tfrac{9}{5}C = (F - 32)$$

$$\tfrac{9}{5}C = F - 32$$

Add 32 to both sides:

$$\tfrac{9}{5}C + 32 = F - 32 + 32$$

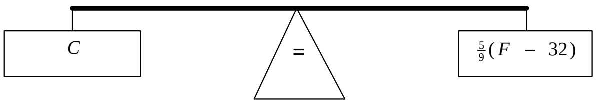
$$\tfrac{9}{5}C + 32 = F$$

Swap sides:

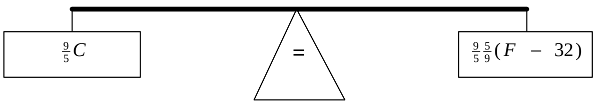
$$F = \tfrac{9}{5}C + 32$$

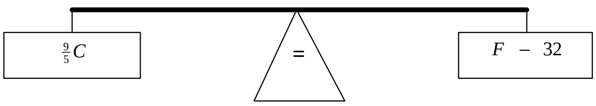
... and with pictures:

Formula:

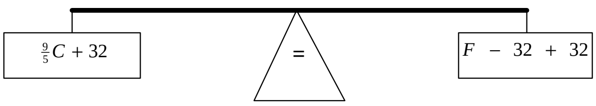


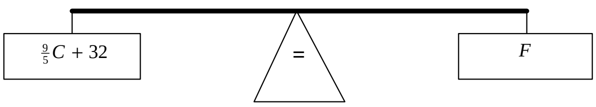
Multiply both sides by $\tfrac{9}{5}$:



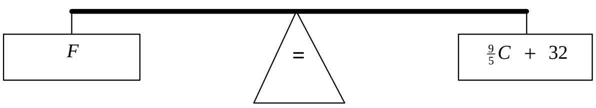


Add 32 to both sides:





Swap sides:



In practice, we need not include as much detail. For such a simple formula we might simply set down the following lines:

$$C = \tfrac{5}{9}(F - 32)$$

Multiply both sides by $\tfrac{9}{5}$: $\tfrac{9}{5}C = F - 32$

Add 32 to both sides: $\tfrac{9}{5}C + 32 = F$

Swap sides: $F = \tfrac{9}{5}C + 32$.

This abbreviated version may give the impression that terms are moving around the equation, but they are not. You must always keep in mind what is truly happening in the background. Always think BALANCE.

<u>Further Examples</u>

Note: In the following examples, full details are shown. In practice, the level of detail you show in your own working will depend on your own confidence and ability; as these grow, you will naturally display less.

E6. For the formula $y = 3 + 4x^2$ make x the subject.
[Note: This is the same as saying, solve for x .]

Analyse operations. Given *x*, how is *y* calculated?

- square
- multiply by 4
- add 3

Reverse and invert operations:

- subtract 3
- divide by 4
- square root

Apply inverse operations to formula:

Formula: $y = 3 + 4x^2$

Subtract 3 :
$$y - 3 = 3 + 4x^2 - 3$$
$$y - 3 = 4x^2$$

Divide by 4 :
$$\frac{y - 3}{4} = \frac{4x^2}{4}$$
$$\frac{y - 3}{4} = x^2$$

At this point it is useful to swap sides

$$x^2 = \frac{y - 3}{4}$$

Square root:
$$\sqrt{x^2} = \sqrt{\frac{y - 3}{4}}$$

$$\underline{\underline{x = \sqrt{\frac{y - 3}{4}}}} \; .$$

Shortened version (for the more confident):

Formula: $y = 3 + 4x^2$

Subtract 3 : $y - 3 = 4x^2$

Swap sides and divide by 4 : $x^2 = \dfrac{y - 3}{4}$

Square root: $\underline{\underline{x = \sqrt{\dfrac{y - 3}{4}}}} \; .$

E7. For the formula $V = \frac{4}{3}\pi r^3$, make r the subject.

Analyse operations. Given r, how is V calculated?

- cube (or raise to the power 3)
- multiply by $\frac{4\pi}{3}$

Reverse and invert operations:

- multiply by $\frac{3}{4\pi}$
- cube root (or raise to the power $1/3$)

Apply inverse operations to formula:

Formula: $V = \frac{4}{3}\pi r^3$

$V = \frac{4\pi}{3} r^3$

Multiply by $\frac{3}{4\pi}$: $\frac{3}{4\pi} V = \frac{3}{4\pi}\frac{4\pi}{3} r^3$

$\frac{3V}{4\pi} = r^3$

Swap sides: $r^3 = \frac{3V}{4\pi}$

Raise to the power $1/3$: $\left(r^3\right)^{1/3} = \left(\frac{3V}{4\pi}\right)^{1/3}$

$\underline{\underline{r = \left(\frac{3V}{4\pi}\right)^{1/3}}}.$

Shortened version:

Formula: $V = \frac{4\pi}{3} r^3$

Swap sides and multiply by $\frac{3}{4\pi}$: $r^3 = \frac{3V}{4\pi}$

Raise to the power $1/3$: $\underline{\underline{r = \left(\frac{3V}{4\pi}\right)^{1/3}}}.$

E8. For the equation $y = a\,e^{6x}$, solve for x .

Note: For BODMAS purposes, you have to imagine brackets grouping the power terms together, i.e. $y = a\,e^{(6x)}$

Analyse operations. Given x, how is y calculated?

- multiply by 6
- e to the power (i.e. $e^{(\ )}$)
- multiply by a

Reverse and invert operations:

- divide by a
- take natural log (i.e. $\ln(\ \ )$)
- divide by 6

Apply inverse operations to formula:

Equation: $$y = a\,e^{6x}$$

Divide by a : $$\frac{y}{a} = \frac{a\,e^{6x}}{a}$$

$$\frac{y}{a} = e^{6x}$$

Natural log: $$\ln\left(\frac{y}{a}\right) = \ln\left(e^{6x}\right)$$

$$\ln\left(\frac{y}{a}\right) = 6x$$

Swap sides: $$6x = \ln\left(\frac{y}{a}\right)$$

Divide by 6 : $$\frac{6x}{6} = \frac{1}{6}\ln\left(\frac{y}{a}\right)$$

$$x = \frac{1}{6}\ln\left(\frac{y}{a}\right).$$

Shortened version:

Equation: $y = a\,e^{6x}$

Divide by a : $\dfrac{y}{a} = e^{6x}$

Swap sides and take natural log: $6x = \ln\left(\dfrac{y}{a}\right)$

Divide by 6 : $\underline{\underline{x = \dfrac{1}{6}\ln\left(\dfrac{y}{a}\right)}}.$

<u>Examples (with less detail)</u>

E9. Determine t when $10^{0.5t} = 8$.

This is not a formula with a subject, so we cannot write down a sequence of operations. It is, however, an equation (with an unknown) and the concept of balance still applies.

In manipulating this equation, we want to end up with the form $t = (\ \ )$. Clearly we must manipulate t down to the level of the equals sign. One of the laws of logarithms will help here:

Equation: $10^{0.5t} = 8$

Take log (base 10) of both sides: $\log_{10}\left(10^{0.5t}\right) = \log_{10}(8)$

$$0.5t = \log_{10}(8)$$

Multiply both sides by 2 : $2 \times 0.5t = 2\log_{10}(8)$

$$t = 2\log_{10}(8)$$

$$\underline{\underline{t \approx 1.8062}}\,.$$

E10. Determine t when $e^{-4t} = 0.5$.

Equation: $e^{-4t} = 0.5$

Take natural log of both sides: $\ln\left(e^{-4t}\right) = \ln(0.5)$

$-4t = \ln(0.5)$

Divide both sides by -4 :
(same as multiplying by $-\frac{1}{4}$)

$t = -\frac{1}{4}\ln(0.5)$

$t \approx 0.1733$.

E11. Determine x when $2^{4x} = 6$.

Equation: $2^{4x} = 6$

Take log (base 10) of both sides: $\log_{10}\left(2^{4x}\right) = \log_{10}(6)$

$4x\log_{10}(2) = \log_{10}(6)$

Divide both sides by $4\log_{10}(2)$: $x = \dfrac{\log_{10}(6)}{4\log_{10}(2)}$

$x \approx 0.6462$.

Note: You could use natural logs and get the same answer.

Applied Examples

E12. The current (i) in the branch of an electronic circuit changes with time (t) in line with the formula

$$i = i_0 e^{-kt} .$$

The initial current (i.e. when $t = 0$) is 15 mA. It takes 4.7 s for the current to drop to 7.5 mA (i.e. half its initial value).

(i). From the numerical information given, determine the values of the parameters i_0 and k.

(ii). Determine the current when $t = 6.5$ s.

(iii). Determine t when the current is 25% of its initial value.

(i). When $t = 0$, $i = 15$ (Note: As long as we are consistent, we don't need to convert mA to A). Input these values into the formula:

$$15 = i_0 e^{-k.0}$$

$$15 = i_0 \, . \, 1$$

$$\underline{\underline{i_0 = 15}} \qquad \rightarrow \qquad i = 15e^{-kt} .$$

When $t = 4.7$, $i = 7.5$. Input these values into the updated formula and solve for k:

$$7.5 = 15e^{-4.7k}$$

$$e^{-4.7k} = 0.5$$

$$\ln\left(e^{-4.7k}\right) = \ln(0.5)$$

$$-4.7k = \ln(0.5)$$

$$k = \frac{\ln(0.5)}{-4.7} \approx \underline{\underline{0.147478123}} .$$

Completed formula is: $i = 15e^{-0.14748t}$; we use this to answer the remaining parts.

(ii). Set $t = 6.5$ in formula and evaluate i :

$$i = 15e^{-0.14748 \times 6.5} = \underline{\underline{5.75 \text{ mA}}} .$$

(iii). 25% of $15 = 3.75$; set $i = 3.75$ in formula and solve for t :

$$3.75 = 15e^{-0.14748\, t}$$

$$e^{-0.14748\, t} = 0.25$$

$$\ln\left(e^{-0.14748\, t}\right) = \ln(0.25)$$

$$-0.14748\, t = \ln(0.25)$$

$$t = \frac{\ln(0.25)}{-0.14748} = \underline{\underline{9.40 \text{ s}}} .$$

E13. The decay of the radioactive element radium is modelled by the formula

$$A = A_0\,(0.5)^{t/1620}$$

where A_0 is the initial amount of radium and A is the amount remaining after t years.

(i). How much radium remains in a 1 kg sample after 1000 years?

(ii). How long would it take for a 1 kg sample to decay to 0.01 kg?

(i). Set $A_0 = 1$, $t = 1000$ in formula and evaluate:

$$A = (0.5)^{1000/1620} = \underline{\underline{0.6519 \text{ kg}}} .$$

(ii). Set $A_0 = 1$, $A = 0.01$ in formula and solve for t :

$$0.01 = (0.5)^{t/1620}$$

$$\ln(0.5)^{t/1620} = \ln(0.01)$$

$$\frac{t}{1620}\ln(0.5) = \ln(0.01)$$

$$t = \frac{1620\,\ln(0.01)}{\ln(0.5)} = \underline{\underline{10763 \text{ years}}} .$$

E14. In the formula

$$R = \frac{R_1 R_2}{R_1 + R_2}$$

make R_1 the subject.

Because the new subject appears in more than one position in the formula, we cannot deconstruct the formula into a sequence of operations that reveal how it should be manipulated. This means we have to try and determine the appropriate operations as we go. Ultimately, we want to achieve to form

$$R_1 = \text{(expression with } R \text{ and } R_2 \text{ only)} \quad ,$$

so we must manipulate the equation in such a way that the R_1's are combined. Fractions can be awkward to handle, so let us get rid of the fraction.

Multiply both sides of the formula by the denominator $R_1 + R_2$:

$$(R_1 + R_2)\, R = (R_1 + R_2) \frac{R_1 R_2}{R_1 + R_2}$$

This simplifies to

$$(R_1 + R_2)\, R = R_1 R_2 \quad .$$

Now multiply out the brackets:

$$R_1 R + R_2 R = R_1 R_2 \quad .$$

Subtract $R_1 R$ from both sides:

$$R_1 R + R_2 R - R_1 R = R_1 R_2 - R_1 R$$

$$R_2 R = R_1 R_2 - R_1 R \quad .$$

R_1 now appears on one side of the equation and at the same level. Swap sides and take out a common factor:

$$R_1 (R_2 - R) = R_2 R \quad .$$

Now divide both sides by $R_2 - R$ to give the desired result:

$$\underline{\underline{R_1 = \frac{R_2 R}{R_2 - R}}} \quad .$$

See over for the shortened version with less detail:

Formula: $$R = \frac{R_1 R_2}{R_1 + R_2}$$

Manipulations: $$(R_1 + R_2) R = R_1 R_2 \ .$$

$$R_1 R + R_2 R = R_1 R_2 \ .$$

$$R_2 R = R_1 R_2 - R_1 R \ .$$

$$R_1 (R_2 - R) = R_2 R \ .$$

$$\underline{\underline{R_1 = \frac{R_2 R}{R_2 - R}}} \ .$$

The above examples illustrate the process of algebraic manipulation, but every possible twist and turn cannot be covered. You must now develop your understanding of the process through practice and be prepared to wrestle with problems to achieve a result.

7. Linear Equations – A Reminder

A linear equation in one variable has the form

$$ax = b$$

where a and b are known constants. To solve this equation for x we must find a value for the variable x that will make the equation true.

7.1. Procedure for Solving Linear Equations

- If the equation contains any fractions multiply both sides of the equation by the least common multiple (LCM) to eliminate the fractions.

- If the denominators of the fractions contain any variables identify the values of these variables that result in division by zero as they must be avoided in the final solution.

- Simplify the equation by removing any brackets and combining like terms.

- Transpose the equation so that terms involving the unknown variable appear on one side of the equality with the remaining terms on the other side.

- Apply addition and subtraction as necessary to combine terms and simplify the equation to the form $ax = b$.

- Isolate the unknown variable (x) by either dividing both sides of the equation by the coefficient a, if a is an integer, or multiplying both sides of the equation by a, if it is a fraction.

- Check your answer by substituting the solution into the original equation.

Examples

E15. **(i).** Solve the equation $2(4x + 1) = 7(x + 3)$.

Solution

$2(4x + 1) = 7(x + 3)$

$8x + 2 = 7x + 21$ (expand the brackets)

$8x - 7x = 21 - 2$ (gather like terms)

$\underline{\underline{x = 19}}$. (simplify)

(ii). Solve the equation, $\frac{4x + 2}{3} + \frac{2x + 1}{2} = \frac{1}{6}$.

$\frac{4x + 2}{3} + \frac{2x + 1}{2} = \frac{1}{6}$

$\frac{6(4x + 2)}{3} + \frac{6(2x + 1)}{2} = \frac{6}{6}$ (multiply through by the LCM of 2, 3,and 6, i.e. 6)

$2(4x + 2) + 3(2x + 1) = 1$

$8x + 4 + 6x + 3 = 1.$ (expand brackets)

$14x = -6$ (gather like terms and simplify)

$x = -\frac{6}{14} \Rightarrow \underline{\underline{x = -\frac{3}{7}}}$. (divide both sides by 14, i.e. the coefficient of *x*)

8. Quadratic Equations – A Reminder

A quadratic equation in x has the form,

$$a x^2 + b x + c = 0 .$$

where a, b and c are constants, with $a \neq 0$, and x is the unknown which we wish to solve for. This type of equation is solved either by factorisation or by use of the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} .$$

Examples

E16. (i). Solve $x^2 + 2x - 8 = 0$ by factorisation.

$$x^2 + 2x - 8 = 0$$

$$(x + 4)(x - 2) = 0$$

$$\left.\begin{matrix} (x + 4) = 0 \\ \text{or} \\ (x - 2) = 0 \end{matrix}\right\} \rightarrow \left.\begin{matrix} x = -4 \\ \text{or} \\ x = +2 \end{matrix}\right\}$$

(ii). Solve $x^2 + 2x - 8 = 0$ by the quadratic formula.

$$x^2 + 2x - 8 = 0 \quad \rightarrow \quad a = 1 \ , \ b = 2 \ , \ c = -8$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times (-8)}}{2 \times 1}$$

$$= \frac{-2 \pm \sqrt{36}}{2}$$

$$= \frac{-2 \pm 6}{2}$$

$$= \frac{-8}{2} \quad \text{or} \quad \frac{4}{2}$$

$$= \underline{\underline{-4 \quad \text{or} \quad 2}}$$

Note: Quadratic equations may have two real solutions, one real solution or no real solutions, depending on the value of the discriminant $\sqrt{b^2 - 4ac}$.

9. Simultaneous Linear Equations – A Reminder

Equations can contain more than one unknown. For example, the equation

$$x + 2y = -1$$

has two unknowns. A solution of this equation is made up of an *x*-value and a *y*-value that together satisfy the equation. We could have $(x = -1, y = 0)$ or $(x = 5, y = -3)$ or any one of an infinite number of solutions. When plotted on a Cartesian axes system, all possible solutions of this equation lie on a straight line. If we have a second equation, say,

$$4x - 3y = 18,$$

this also has an infinite number of solutions, which also lie on a straight line. However, there is one solution that is common to both equations. Geometrically, this common solution is given by the coordinates of the point of intersection of the two straight-line graphs.

There are two algebraic methods to solve simultaneous linear equations.

Method 1 – Elimination by Substitution

1) Choose an equation and express one of the variables *x*, say, in terms of the other variable, *y*.
2) The expression for *x* is then substituted into the other equation to form a single equation in a single unknown which can be solved for *y*.
3) Substitute for *y* in the expression for *x*, found at Step 1, to obtain the full solution.

E17.(i). Consider the equations $x + 2y = -1$ (1)

$4x - 3y = 18$ (2)

Rearrange equation (1) to give, $x = -1 - 2y$.

Substitute for *x* into equation (2) to give

$$4(-1 - 2y) - 3y = 18$$

$$-4 - 8y - 3y = 18$$

$$-11y = 22$$

$$y = -2$$

Now substitute $y = -2$ in the expression obtained earlier for *x*, i.e. $x = -1 - 2y$.

We have $x = -1 - 2(-2) = -1 + 4 = 3$.

Hence solutions are $\underline{\underline{x = 3}}$, $\underline{\underline{y = -2}}$.

These are the **unique** pair of values that satisfy both equations (1) and (2) at the same time.

Alternatively, we could have rearranged equation (1) for *y* to obtain $y = -\frac{1}{2} - \frac{1}{2}x$, which is substituted into equation (2) to find $x = 3$. Then substitute $x = 3$ in the expression for *y* to obtain $y = -2$.

Method 2 – Elimination by the Addition or Subtraction of Equations

1) Make the coefficients of *x* or *y* the same in both equations.

2) Add or subtract the equations in Step 1 to eliminate one variable and give one equation in one unknown which can easily be solved.

3) The variable value obtained in Step 2 is then substituted in either of the original equations to obtain the full solution.

(ii). Once again consider the equations:

$$x + 2y = -1 \qquad (1)$$

$$4x - 3y = 18 \qquad (2)$$

Step 1: Make the coefficient of *y* the same in both equations.

Multiply equation (1) by 3: $3x + 6y = -3$ (3)

Multiply equation (2) by 2: $8x - 6y = 36$ (4)

Step 2: Eliminate *y* and solve the resulting one variable equation for *x*.

Add (3) to (4): $11x = 33 \Rightarrow x = 3$

Step 3: Substitute for *x* in either equation (1) or (2) and solve for *y*.

Substitute $x = 3$ in equation (1) to obtain $y = -2$.

Hence solutions are $\underline{\underline{x = 3}}$, $\underline{\underline{y = -2}}$.

Whichever method is used the solution should be checked by substituting the values for x and y into **both** the original equations.

$$\begin{aligned} \text{1st equation :} \quad \text{LHS} &= x + 2y \\ &= 3 + 2 \times (-2) \\ &= -1 \\ &= \text{RHS} \end{aligned}$$

$$\begin{aligned} \text{2nd equation :} \quad \text{LHS} &= 4x - 3y \\ &= 4 \times 3 - 3 \times (-2) \\ &= 12 + 6 \\ &= 18 \\ &= \text{RHS} . \end{aligned}$$

We note here that when we have two linear equations in two unknowns there are three possibilities:

- the lines intersect at a point – unique solution
- the lines are parallel – no solution
- the lines are coincident – infinitely many solutions.

Tutorial Exercises

1. Expansion of Algebraic Expressions Containing Brackets (Revision)

Q1.1. Expand the following expressions involving products and powers:

(i). $(3x^2)(5x^4)$ **(ii).** $(8x^3)(9x^5)$

(iii). $(4xy)(7x^2y^3)$ **(iv).** $(-3x^3y^4)(6x^8y^9)$

(v). $(5x^2y^7)(-7x^3y^9)$ **(vi).** $(3ab)^2(-5ab)$

(vii). $(-5x^2y)(-8x^3y^2)$ **(viii).** $(-3a^2b)(-4ab^3)$

(ix). $(4a)(ab)(-a^2b)$ **(x).** $(-3ab)(-4a^2b)(ab^2)^2$.

Q1.2. Open out the following bracketed expressions:

(i). $(x+2)(x+3)$ **(ii).** $(2x+5)(3x-7)$

(iii). $(2x-3)(2x+3)$ **(iv).** $(4y-8)(6y-13)$

(v). $(x+4)^2$ **(vi).** $(2x+3y)^2$

(vii). $(2x-3y)^2$ **(viii).** $(x+2)(x^2-4x+5)$

(ix). $(2x-1)(x^2-3x+7)$ **(x).** $(x+3)(x-4)(x+5)$.

Q1.3. Factorise the following quadratic expressions:

(i). x^2+3x+2 **(ii).** x^2+5x+4

(iii). x^2+4x+4 **(iv).** x^2+x-2

(v). x^2-x-2 **(vi).** x^2+5x-6

(vii). x^2-5x-6 **(viii).** x^2+x-6

(ix). x^2-x-6 **(x).** $2x^2+11x+12$.

Q1.4. Show that $(a+b)(a-b) = a^2 - b^2$ and use the result to

(i). expand: $(x+2)(x-2)$; $(2x-3)(2x+3)$

(ii). factorise: $x^2 - 16$; $25x^2 - 64$.

2. Division of Algebraic Expressions

Q2.1. Perform the following divisions:

(i). $\dfrac{12a^2}{3a^2}$ **(ii).** $\dfrac{-32b^3}{8b^2}$

(iii). $\dfrac{42a^3b^4}{6a^2b}$ **(iv).** $\dfrac{72c^2d^3}{-9cd^2}$

(v). $\dfrac{4xy - 8xy^2}{2xy}$ **(vi).** $\dfrac{6\pi rh^2 + 18\pi r^2h}{3\pi rh}$.

Q2.2. Use polynomial division to express the following quotients in the form

$$\text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}} .$$

(i). $\dfrac{x^2 - 7x + 2}{x - 3}$ **(ii).** $\dfrac{6x^2 + 7x + 5}{3x + 5}$

(iii). $\dfrac{16x^2 - 1}{2x - 1}$ **(iv).** $\dfrac{x^3 - 1}{x - 1}$

(v). $\dfrac{x^3 - 7x^2 + 5x + 2}{x^2 - x - 2}$ **(vi).** $\dfrac{4x^3 - 2x - 1}{2x^2 + 1}$.

3. Exponentials and Logarithms – The Basics

Q3.1. Use your calculator functions, namely $\log$ (which is actually $\log_{10}$) and 10^x, to evaluate the following expressions. Comment on the results for parts (v) – (viii).

(i). $\log_{10}(1000)$ **(ii).** $\log_{10}(2.5)$

(iii). $10^{3.5}$ **(iv).** $10^{-1.4}$

(v). $\log_{10}(10^{3.5})$ **(vi).** $\log_{10}(10^{-1.4})$

(vii). $10^{\log_{10}(6.1)}$ **(viii).** $10^{\log_{10}(0.5)}$.

Q3.2. Use your calculator functions, namely $\ln$ (which is actually $\log_e$) and e^x, to evaluate the following expressions. Comment on the results for parts (v) – (viii).

(i). $\ln(4.5)$ **(ii).** $\ln(2.75)$

(iii). $e^{0.6}$ **(iv).** $e^{-1.5}$

(v). $\ln(e^{0.6})$ **(vi).** $\ln(e^{-1.5})$

(vii). $e^{\ln(2.5)}$ **(viii).** $e^{\ln(0.75)}$.

Q3.3. Write each of the following expressions as sums and differences of logarithms (where possible, without using powers):

(i). $\log_{10}\left(\dfrac{3x^2}{y}\right)$ **(ii).** $\ln\left(\dfrac{x^2 y^2}{4}\right)$

(iii). $\log_{10}\left(\dfrac{100}{x+1}\right)$ **(iv).** $\ln\left(\dfrac{e^2}{2x+3}\right)$

(v). $\log_{10}\sqrt{x^2+1}$ **(vi).** $\ln\sqrt{\dfrac{(x+1)^3(x-1)}{(x+2)}}$.

Q3.4. Write each of the following as a single logarithm:

(i). $3\log_{10} x + 2\log_{10} y - 4\log_{10} z$ **(ii).** $2\log_{10}(x+y) - \tfrac{1}{2}\log_{10} z$

(iii). $3\ln x + \tfrac{1}{3}\ln y$ **(iv).** $4\ln(2x+y) - 2\ln z$.

4 Changing the Subject of a Formula

Q4.1. For each of the following formulae, change the subject to the quantity indicated in brackets:

(i). $v = u + at$ (t)

(ii). $s = \frac{1}{2}at^2$ (t)

(iii). $s = ut + \frac{1}{2}at^2$ (u)

(iv). $s = ut + \frac{1}{2}at^2$ (a)

(v). $v = \sqrt{u^2 + 2as}$ (s)

(vi). $v = \sqrt{u^2 + 2as}$ (u)

(vii). $y = a + bx^3$ (x)

(viii). $i = 5e^t$ (t)

(ix). $i = 8e^{-2t}$ (t)

(x). $y = 10^{(2x+1)}$ (x)

(xi). $y = 10^{(3x-2)}$ (x)

(xii). $y = c_0 + a_0 e^{-kt}$ (t) .

Q4.2. For each of the following formulae, change the subject to the quantity indicated in brackets:

(i). $y = \dfrac{2 - x}{3 + x}$ (x) **(ii).** $y = \dfrac{4 + 2x}{5 - x}$ (x)

(iii). $z = \dfrac{xy}{x + y}$ (y) **(iv).** $C = \dfrac{C_1 C_2}{C_1 - C_2}$ (C_2) .

5. Solution of Equations

Q5.1. Solve the following equations:

(i). $4x + 5 = 8$

(ii). $10x - 8 = -12$

(iii). $6x + 3 = 2x - 5$

(iv). $3x - 9 = 5x + 2$

(v). $5 + 3x^2 = 32$

(vi). $5x^3 + 320 = 0$

(vii). $e^x = 0.75$

(viii). $e^{4x} = 0.2$

(ix). $5e^{-x} - 8 = 7$

(x). $4 + 12e^{-3x} = 13$

(xi). $\ln(3x) = -2$

(xii). $5\ln(2x) + 3 = 0$

(xiii) $5 - 3\ln(4x) = -10$

(xiv). $\ln(6x+4) = 0.25$

(xv). $10^x = 2.5$

(xvi). $10^{3x-2} = 1.75$

(xvii). $\log_{10}(4x) = 0.32$

(xviii). $\log_{10}(5x+4) = 0.65$

(xix). $2^{4x-1} = 5$

(xx). $3^{2x+4} = 6$

Q5.2. Solve the following quadratic equations by factorisation, then repeat using the quadratic formula:

(i). $x^2 + 2x - 15 = 0$

(ii). $x^2 - 9x + 20 = 0$

(iii). $x^2 + 10x + 25 = 0$

(iv). $2x^2 - 7x - 4 = 0$.

Q5.3. The following quadratic equations do not have any real solutions. Try solving them by the quadratic formula and see what happens:

(i). $x^2 + 2x + 2 = 0$ **(ii).** $2x^2 - 3x + 4 = 0$.

Q5.4. Solve the following sets of simultaneous equations:

(i). $\left.\begin{aligned} 4x + 3y &= 2 \\ 2x - y &= 16 \end{aligned}\right\}$ **(ii).** $\left.\begin{aligned} 5x + 2y &= 1 \\ 4x + 3y &= -2 \end{aligned}\right\}$

(iii). $\left.\begin{aligned} 6x - 2y &= -20 \\ 4x + 5y &= -7 \end{aligned}\right\}$ **(iv).** $\left.\begin{aligned} 8x - 5y &= 24.5 \\ 2x - 3y &= 10.5 \end{aligned}\right\}$.

Answers to Tutorial Questions

A1.1. (i). $15x^6$ **(ii).** $72x^8$

(iii). $28x^3 y^4$ **(iv).** $-18x^{11} y^{13}$

(v). $-35x^5 y^{16}$ **(vi).** $-45a^3 b^3$

(vii). $40x^5 y^3$ **(viii).** $12a^3 b^4$

(ix). $-4a^4 b^2$ **(x).** $12a^5 b^6$

A1.2. (i). $x^2 + 5x + 6$ **(ii).** $6x^2 + x - 35$

(iii). $4x^2 - 9$ **(iv).** $24y^2 - 100y + 104$

(v). $x^2 + 8x + 16$ **(vi).** $4x^2 + 12xy + 9y^2$

(vii). $4x^2 - 12xy + 9y^2$ **(viii).** $x^3 - 2x^2 - 3x + 10$

(ix). $2x^3 - 7x^2 + 17x - 7$ **(x).** $x^3 + 4x^2 - 17x - 60$

A1.3. (i). $(x+1)(x+2)$ **(ii).** $(x+4)(x+1)$

(iii). $(x+2)^2$ **(iv).** $(x+2)(x-1)$

(v). $(x-2)(x+1)$ **(vi).** $(x+6)(x-1)$

(vii). $(x-6)(x+1)$ **(viii).** $(x+3)(x-2)$

(ix). $(x-3)(x+2)$ **(x).** $(2x+3)(x+4)$

A1.4. (i). $(x+2)(x-2) = x^2 - 4$; $(2x-3)(2x+3) = 4x^2 - 9$

(ii). $x^2 - 16 = (x+4)(x-4)$; $25x^2 - 64 = (5x+8)(5x-8)$

A2.1. (i). 4 **(ii).** $-4b$

(iii). $7ab^3$ **(iv).** $-8cd$

(v). $2 - 4y$ **(vi).** $2h + 6r$

A2.2. **(i).** $(x-4) - \dfrac{10}{x-3}$ **(ii).** $(2x-1) + \dfrac{10}{3x+5}$

(iii). $(8x+4) + \dfrac{3}{2x-1}$ **(iv).** $x^2 + x + 1$ [no remainder]

(v). $(x-6) + \dfrac{x-10}{x^2 - x - 2}$ **(vi).** $2x - \dfrac{4x+1}{2x^2+1}$

A3.1. **(i).** 3 **(ii).** 0.397940008

(iii). 3162.27766 **(iv).** 0.039810717

(v). 3.5 **(vi).** –1.4

(vii). 6.1 **(viii).** 0.5

A3.2. **(i).** 1.504077397 **(ii).** 1.011600912

(iii). 1.822118800 **(iv).** 0.223130160

(v). 0.6 **(vi).** –1.5

(vii). 2.5 **(viii).** 0.75

A3.3. **(i).** $\log_{10} 3 + 2\log_{10} x - \log_{10} y$ **(ii).** $2\ln x + 2\ln y - \ln 4$

(iii). $2 - \log_{10}(x+1)$ **(iv).** $2 - \ln(2x+3)$

(v). $\frac{1}{2}\log_{10}(x^2+1)$

(vi). $\frac{1}{2}[3\ln(x+1) + \ln(x-1) - \ln(x+2)]$

A3.4. **(i).** $\log_{10}\left(\dfrac{x^3 y^2}{z^4}\right)$ **(ii).** $\log_{10}\left[\dfrac{(x+y)^2}{\sqrt{z}}\right]$

(iii). $\ln\left(x^3 y^{1/3}\right)$ **(iv).** $\ln\left[\dfrac{(2x+y)^4}{z^2}\right]$

Q4.1. (i). $t = \dfrac{v - u}{a}$ **(ii).** $t = \sqrt{\dfrac{2s}{a}}$

(iii). $u = \dfrac{s - \frac{1}{2}at^2}{t}$ **(iv).** $a = \dfrac{2(s - ut)}{t^2}$

(v). $s = \dfrac{v^2 - u^2}{2a}$ **(vi).** $u = \sqrt{v^2 - 2as}$

(vii). $x = \left(\dfrac{y - a}{b}\right)^{1/3}$ **(viii).** $t = \ln\left(\dfrac{i}{5}\right)$

(ix). $t = -\frac{1}{2}\ln\left(\dfrac{i}{8}\right)$ **(x).** $x = \frac{1}{2}[\log_{10}(y) - 1]$

(xi). $x = \frac{1}{3}[\log_{10}(y) + 2]$ **(xii).** $t = -\dfrac{1}{k}\ln\left(\dfrac{y - c_0}{a_0}\right)$

Q4.1. (i). $x = \dfrac{2 - 3y}{y + 1}$ **(ii).** $x = \dfrac{5y - 4}{y + 2}$

(iii). $y = \dfrac{xz}{x - z}$ **(iv).** $C_2 = \dfrac{C\,C_1}{C + C_1}$

Q5.1. (i). $x = \frac{3}{4}$ **(ii).** $x = -\frac{2}{5}$

(iii). $x = -2$ **(iv).** $x = -\frac{11}{2}$

(v). $x = +3 \quad , \quad x = -3$ **(vi).** $x = 4$

(vii). $x = \ln(0.75) = -0.28768$ **(viii).** $x = \frac{1}{4}\ln(0.2) = -0.40236$

(ix). $x = -\ln(3) = -1.09861$ **(x).** $x = -\frac{1}{3}\ln(0.75) = 0.09589$

(xi). $x = \frac{1}{3}e^{-2} = 0.04511$ **(xii).** $x = \frac{1}{2}e^{-0.6} = 0.27441$

(xiii). $x = \frac{1}{4}e^{5} = 37.10329$ **(xiv).** $x = \frac{1}{6}(e^{0.25} - 4) = -0.45266$

(xv). $x = \log_{10}(2.5) = 0.39794$ **(xvi).** $x = \frac{1}{3}[\log_{10}(1.75) + 2] = 0.74768$

(xvii). $x = \frac{1}{4} \cdot 10^{0.32} = 0.52232$ **(xviii).** $x = \frac{1}{5}[10^{0.65} - 4] = 0.09337$

(xix). / Over the page . . .

(xix). $x = \frac{1}{4}\left[\frac{\log_{10} 5}{\log_{10} 2} + 1\right] = 0.83048$ [could also use natural logs]

(xx). $x = \frac{1}{2}\left[\frac{\log_{10} 6}{\log_{10} 3} - 4\right] = -1.18454$ [could also use natural logs]

Q5.2. **(i).** $x = -5$, $x = +3$ **(ii).** $x = +4$, $x = +5$

(iii). $x = -5$ [repeated root] **(iv).** $x = -\frac{1}{2}$, $x = +4$

Q5.3. Negative values under the square root sign indicate no real solutions. Solutions are only possible by moving to complex numbers.

Q5.4. **(i).** $x = 5$, $y = -6$ **(ii).** $x = 1$, $y = -2$

(iii). $x = -3$, $y = 1$ **(iv).** $x = 1.5$, $y = -2.5$