Laplace Transforms

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Laplace Transforms

1) Introduction

The idea of transforming a “difficult” problem into an “easier” problem is one that is used widely in mathematics. Diagrammatically we have

There are many types of transforms available to mathematicians, engineers and scientists. We are going to examine one such transformation, the Laplace transform, which can be used to solve certain types of differential equations and also has applications in control theory.

2) Definition

The Laplace transform operates on functions of $t$. Given a function $f(t)$ we define its Laplace transform as

$$ L[ f(t) ] = \int_{0}^{\infty} f(t) e^{-st} \, dt $$

where $s$ is termed the Laplace variable. We sometimes denote the Laplace transform by

$$ F(s) \quad \text{or} \quad \overline{f}.$$ 

We start off with a function of $t$ and end up with a function of $s$; $s$ is, in fact, a complex variable, but this need not concern us too much.

Because the function of $t$ is often some form of time signal, we often talk about moving from the time domain to the Laplace domain when we perform a Laplace transformation.

Note: Laplace transforms are only concerned with functions where $t \geq 0$. 
Examples

(1) \( f(t) = 1 : \)

\[
L[1] = \int_0^\infty 1.e^{-st} \, dt
\]

\[
= \left[ -\frac{1}{s}e^{-st} \right]_{t=\infty}^{t=0}
\]

Strictly speaking, we can’t set \( t \) equal to \( \infty \), but we can “take the limit” as \( t \) heads towards infinity. Providing \( s > 0 \), thereby ensuring that we have a negative exponential, the limit of the inside of the square brackets as \( t \) tends to infinity will be zero. Also, since \( e^0 = 1 \), this leaves us with

\[
L[1] = 0 - \left( -\frac{1}{s} \right) = \frac{1}{s}.
\]

So \( f(t) = 1 \rightarrow L[1] = \frac{1}{s} \)

(2) \( f(t) = t : \)

\[
L[t] = \int_0^\infty t.e^{-st} \, dt
\]

This integral requires integration by parts to complete the process, but with the same assumptions regarding \( s \) as before, it can readily be shown that

\[
L[t] = \frac{1}{s^2}.
\]

So \( f(t) = t \rightarrow L[t] = \frac{1}{s^2} \)

Very quickly the integrations required to complete the Laplace transformation become difficult and messy. For this reason, we generally work from a table of pre-determined Laplace transforms (see Appendix).
Examples Using Table of Laplace Transforms

(3) (a) Determine $L[t^3]$.

$t^3$ isn’t in the table explicitly, but $t^{n-1}$ is:

$$L[t^{n-1}] = \frac{(n-1)!}{s^n}$$

For $t^3$ we require $n = 4$:

$$L[t^3] = \frac{3!}{s^4} = \frac{3 \times 2 \times 1}{s^4} = \frac{6}{s^4}$$

(b) Determine $L[e^{-2t}]$.

From table:

$$L[e^{-\alpha t}] = \frac{1}{s + \alpha}.$$  

Set $\alpha = 2$:

$$L[e^{-2t}] = \frac{1}{s + 2}$$

(c) Determine $L[\sin(4t)]$.

From table:

$$L[\sin(\omega t)] = \frac{\omega}{s^2 + \omega^2}.$$  

Set $\omega = 4$:

$$L[\sin(4t)] = \frac{4}{s^2 + 4^2} = \frac{4}{s^2 + 16}$$
3) Properties of the Laplace Transform

The Laplace transform has several special properties that make it a useful mathematical tool. We consider some of these now.

a) Linearity

Suppose we have two functions along with their respective Laplace transforms:
\[ L[ f(t) ] = F(s) \quad L[ g(t) ] = G(s) \, . \]

The property of linearity means that
\[ L[ a f(t) + b g(t) ] = a F(s) + b G(s) \]
providing \( a \) and \( b \) are constants. This makes the transformation of a string of functions straightforward.

Example

\[ (4) \quad L[ 2 \cos(\omega t) + 3 \sin(\omega t) ] = 2 \frac{s}{s^2 + \omega^2} + 3 \frac{\omega}{s^2 + \omega^2} = \frac{2s}{s^2 + \omega^2} + \frac{3\omega}{s^2 + \omega^2} \]

Warning: The Laplace transform of a product is NOT EQUAL TO the product of the individual Laplace transforms. We have to invoke other properties of the Laplace transform to deal with such.

b) The First Shifting Theorem

Suppose a function \( f(t) \) has the Laplace transform \( F(s) \). It is easily demonstrated that
\[ L[ e^{-\alpha t} f(t) ] = F(s + \alpha) \, . \]

Example

\[ (5) \quad L[ t^3 ] = \frac{6}{s^4} \quad [1] \]

By the first shifting property
\[ L[ e^{-2t} t^3 ] = \frac{6}{(s + 2)^4} \quad [2] \]

To obtain [2] from [1] we merely replace \( s \) by \( s + 2 \).
c) Transformation of Derivatives

As before, denote the Laplace transform (LT) of \( f(t) \) by \( F(s) \). Now consider the LT of the derivative of \( f(t) \), denoted by \( \dot{f}(t) \):

\[
L[\dot{f}(t)] = \int_{0}^{\infty} \dot{f}(t) e^{-st} \, dt .
\]

Integrate by parts (integrating \( \dot{f}(t) \) and differentiating \( e^{-st} \)):

\[
= \left[ f(t) e^{-st} \right]_{0}^{\infty} - \int_{0}^{\infty} f(t)(-s) e^{-st} \, dt
\]
\[
= 0 - f(0) + s \int_{0}^{\infty} f(t) e^{-st} \, dt
\]
\[
= -f(0) + s F(s) .
\]

So

\[
L[\dot{f}(t)] = s F(s) - f(0) .
\]

We have expressed the Laplace transform of a derivative in terms of the Laplace transform of the undifferentiated function. In effect, the Laplace transform has converted the operation of differentiation into the simpler operation of multiplication by \( s \).

In a similar fashion, using repeated integration by parts, we can show that

\[
L[\ddot{f}(t)] = s^2 F(s) - s f(0) - \dot{f}(0) .
\]

This is one of the most important properties of the Laplace transform. The Laplace transform “gets rid of” derivatives; just the thing for solving differential equations!

When we come to solve differential equations using Laplace transforms we shall use the following alternative notation:

\[
L[x] = \bar{x}
\]
\[
L[\dot{x}] = s \bar{x} - x(0)
\]
\[
L[\ddot{x}] = s^2 \bar{x} - s x(0) - \dot{x}(0) .
\]

However, before we can solve differential equations, we need to look at the reverse process of finding functions of \( t \) from given Laplace transforms.
4) Inverse Laplace Transforms

So far, we have looked at how to determine the LT of a function of \( t \), ending up with a function of \( s \). The table of Laplace transforms collects together the results we have considered, and more. When we apply Laplace transforms to solve problems we will have to invoke the inverse transformation. That is, given a Laplace transform \( F(s) \) we will want to determine the corresponding \( f(t) \). In general we have

\[
L^{-1}[F(s)] = \frac{1}{2\pi j} \int_{\gamma-j\infty}^{\gamma+j\infty} F(s) e^{st} \, ds ,
\]

where the evaluation of the integral requires a knowledge of complex analysis, which is too difficult to consider here. Instead, we shall rely on the table of Laplace transforms used in reverse to provide inverse Laplace transforms. This will mean manipulating a given Laplace transform until it looks like one or more entries in the right of the table. The inverse is then determined from the left of the table. The following examples illustrate the main algebraic techniques required. These include completing the square, factorisation and the formation of partial fractions. See separate documents for the details of completing the square and partial fractions.

Examples

(6) Invert the Laplace transform \( \frac{3}{s^6} \).

The closest entry in the table gives:

\[
\frac{(n-1)!}{s^n} \rightarrow t^{n-1} .
\]

Setting \( n = 6 \):

\[
\frac{5!}{s^6} \rightarrow t^5 . \quad \text{[Note: \( 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120 \)], so \( \frac{120}{s^6} \rightarrow t^5 \).}
\]

In the given Laplace transform there is a 3 on the top; we would like there to be a 120 to match the table entry. We can re-write the transform providing we don’t alter its “net value”:

\[
\frac{3}{s^6} = 3 \frac{1}{s^6} = \frac{3}{120} \frac{120}{s^6} = \frac{1}{40} \left[ \frac{120}{s^6} \right] .
\]

The term in the square brackets is now exactly the table entry so we can invert that and simply multiply by the fraction in front:

\[
\frac{3}{s^6} \rightarrow \frac{1}{40} t^5 .
\]
(7) Invert \( \frac{1}{s^2 + 2s + 5} \).

This requires the technique of “completing the square” and a little bit of fine tuning to re-write it in a form that can be inverted from tables. Note that the closest entry in the table gives:

\[
\frac{\omega}{(s + \alpha)^2 + \omega^2} \rightarrow e^{-\alpha t} \sin(\omega t).
\]

Now complete the square in the denominator:

\[
s^2 + 2s + 5 = (s + 1)^2 - 1 + 5 = (s + 1)^2 + 4.
\]

The given transform becomes:

\[
\frac{1}{s^2 + 2s + 5} = \frac{1}{(s + 1)^2 + 4} = \frac{1}{(s + 1)^2 + 2^2}.
\]

This is now very close to the table entry with \( \alpha = 1 \) and \( \omega = 2 \). We would like there to be a 2 on top so “fine tuning” gives

\[
\frac{1}{(s + 1)^2 + 2^2} = \frac{1}{2} \left[ \frac{2}{(s + 1)^2 + 2^2} \right],
\]

where the term in the square brackets is exactly the table entry with \( \alpha = 1 \) and \( \omega = 2 \). Inverting this and multiplying by the fraction gives

\[
\frac{1}{2} \left[ \frac{2}{(s + 1)^2 + 2^2} \right] \rightarrow \frac{1}{2} e^{-t} \sin(2t).
\]

(8)(a) Invert \( \frac{s + 5}{s^2 - 2s - 3} \).

Factorising the denominator and splitting the result into its partial fractions deals with this one. Note that the details of the partial fraction expansion have been omitted.

\[
\frac{s + 5}{s^2 - 2s - 3} = \frac{s + 5}{(s - 3)(s + 1)} = \frac{2}{s - 3} - \frac{1}{s + 1}
\]

Use \( \frac{1}{s + \alpha} \rightarrow e^{-\alpha t} \) twice with \( \alpha = -3 \) and \( \alpha = +1 \) to give

\[
\frac{2}{s - 3} - \frac{1}{s + 1} \rightarrow 2e^{3t} - e^{-t}.
\]
Invert \( \frac{s + 5}{s^2 - 2s - 3} \) by completing the square.

If we failed to notice that the denominator factorised, then completing the square is still an option:

\[
\frac{s + 5}{s^2 - 2s - 3} = \frac{s + 5}{(s - 1)^2 - 1 - 3} = \frac{s + 5}{(s - 1)^2 - 4} = \frac{s + 5}{(s - 1)^2 - 2^2}
\]

The minus sign in the denominator is very significant since it no longer conforms to

\[(s + \alpha)^2 + \omega^2 .\]

Instead we need the following results:

\[
L[e^{-at} \sinh(\beta t)] = \frac{\beta}{(s + \alpha)^2 - \beta^2}, \quad L[e^{-at} \cosh(\beta t)] = \frac{s + \alpha}{(s + \alpha)^2 - \beta^2}
\]

where sinh (pronounced “shine”) and cosh (pronounced “cosh”) are the so-called hyperbolic functions. Fine tuning the Laplace transform a little bit more gives

\[
\frac{s + 5}{(s - 1)^2 - 2^2} = \frac{(s - 1) + 6}{(s - 1)^2 - 2^2} = \left[ \frac{(s - 1)}{(s - 1)^2 - 2^2} \right] + 3 \left[ \frac{2}{(s - 1)^2 - 2^2} \right] .
\]

Inverting now gives

\[
\left[ \frac{(s - 1)}{(s - 1)^2 - 2^2} \right] + 3 \left[ \frac{2}{(s - 1)^2 - 2^2} \right] \rightarrow e^{\beta t} \cosh(2t) + 3e^{\beta t} \sinh(2t) = e^{\beta t} [\cosh(2t) + 3\sinh(2t)] .
\]

By applying the following mathematical identities:

\[
\cosh(\beta t) = \frac{1}{2}[e^{\beta t} + e^{-\beta t}] \quad \text{and} \quad \sinh(\beta t) = \frac{1}{2}[e^{\beta t} - e^{-\beta t}]
\]

we can easily show that the two versions of the answer are equivalent.
5) Using Laplace Transforms to Solve ODEs

We have seen how the Laplace transform of the derivative of a function can be expressed in terms of the Laplace transform of the undifferentiated function. We can use this property to derive solutions to certain types of differential equations. The process is broken down into the following steps:

- **Transform** both sides of the ODE;
- **Substitute** initial values;
- **Solve** for \( \bar{x} \);
- **Manipulate** into a form that can be inverted from tables;
- **Invert** to give the solution of the ODE.

The method is best illustrated by example and we shall need the results from earlier:

\[
L[x] = \bar{x} \\
L[\dot{x}] = s \bar{x} - x(0) \\
L[\ddot{x}] = s^2 \bar{x} - s x(0) - \dot{x}(0).
\]

**Example**

(9) Consider the ODE

\[
\ddot{x} + 2 \dot{x} + 5 x = 8 e^{-3t}
\]

Subject to the initial conditions

\[
x(0) = \dot{x}(0) = 0.
\]

**Transform** both sides of the equation:

\[
[s^2 \bar{x} - s x(0) - \dot{x}(0)] + 2 [s \bar{x} - x(0)] + 5 \bar{x} = \frac{8}{s + 3}.
\]

**Substitute** initial values:

\[
[s^2 \bar{x} - s 0 - 0] + 2 [s \bar{x} - 0] + 5 \bar{x} = \frac{8}{s + 3}
\]

\[
(s^2 + 2s + 5) \bar{x} = \frac{8}{s + 3}.
\]
Solve for $\bar{x}$:

$$
\bar{x} = \frac{8}{(s + 3) \left( s^2 + 2x + 5 \right)}.
$$

**Manipulate** into a form that can be inverted from tables. In this case we form partial fractions, complete the square on the quadratic denominator and finish up with a little “fine tuning” of the last term:

$$
\bar{x} = \left[ \frac{1}{s + 3} \right] - \left[ \frac{s}{(s + 1)^2 + 2^2} \right] + \frac{1}{2} \left[ \frac{2}{(s + 1)^2 + 2^2} \right].
$$

Finally **invert** term-by-term and tidy-up:

$$
x(t) = e^{-3t} - e^{-t} \left[ \cos(2t) - \frac{1}{2} \sin(2t) \right] + \frac{1}{2} e^{-t} \sin(2t)
= e^{-3t} - e^{-t} \left[ \cos(2t) - \sin(2t) \right].
$$

**Further Example**

(10) Solve $\ddot{x} + 25x = 1$ subject to $x(0) = 10$ and $\dot{x}(0) = 0$.

**Transform:**

$$
\left[ s^2 \bar{x} - sx(0) - \dot{x}(0) \right] + 25 \bar{x} = \frac{1}{s}.
$$

**Substitute** initial conditions:

$$
\left[ s^2 \bar{x} - 10s - 0 \right] + 25 \bar{x} = \frac{1}{s}.
$$

**Solve** for $\bar{x}$:

$$
s^2 \bar{x} + 25 \bar{x} = \frac{1}{s} + 10s
\Rightarrow (s^2 + 25) \bar{x} = \frac{1}{s} + 10s
\Rightarrow \bar{x} = \frac{1}{s(s^2 + 25)} + 10 \frac{s}{(s^2 + 25)}.
$$
Manipulate and invert:

\[ \vec{x} = \frac{1}{25} \begin{bmatrix} \frac{25}{s(s^2 + 25)} \\ \frac{s}{s(s^2 + 25)} \end{bmatrix} + 10 \begin{bmatrix} \frac{s}{s(s^2 + 25)} \end{bmatrix} \]

\[ x(t) = \frac{1}{25}[1 - \cos(5t)] + 10\cos(5t) \]

Using Laplace transforms to solve ODEs of the form

\[ a \ddot{x} + b \dot{x} + c x = f(t) \]

allows us to tackle problems where a solution by the method of undetermined coefficients (say) is rendered difficult or impossible because of the specific type of function on the right-hand-side. We shall now look at two such functions for which this is the case.

6) The Step Function

The step function is defined as

\[ u(t - T) = \begin{cases} 
0 & , \ t < T \\
1 & , \ t > T
\end{cases} \]

Its graph is shown below in Figure 1:

\[ T \]

is called the **critical value** of the step function; it is where the function changes value. The value of the step function at \( t = T \) is not defined above. Some authors define the value as 1, others define it as 0 or 0.5. We shall not worry about it!

**Note:** In the notation for the step function, the \( u \) is NOT multiplying the bracket. The \( u \) is the short-hand name for the function (full name: unit step function) and the bracket contains information regarding the function variable (usually \( t \)) and the location of the step.
The step function is like a mathematical switch. It can be used to model situations where something occurs suddenly, such as when a circuit is switched on, or a voltage is instantaneously modified, or a force is suddenly applied or removed. This may be done by taking combinations of step functions with different critical values, or even combining them with more conventional functions, as the following examples illustrate.

Note: Since Laplace transforms are only concerned with functions where \( t \geq 0 \), if the step occurs at \( t = 0 \) then \( u(t - 0) \equiv 1 \).

Examples

(11) Graph \( f(t) = 2u(t - 5) - 3u(t - 10) + u(t - 15) \).

Thinking of the step functions as switches, for \( t < 5 \) all step functions are “off” and so are all zero; for \( 5 < t < 10 \) the first step function is “on” and has the value 1; for \( 10 < t < 15 \) the second step function now switches on to 1 also; and for \( t > 15 \) all step functions are “on” and equal to 1. This gives

\[
f(t) = \begin{cases} 
0, & 0 < t < 5 \\
2, & 5 < t < 10 \\
-1, & 10 < t < 15 \\
0, & t > 15 
\end{cases}
\]

and the graph

![Graph of f(t)](image)

(12) Over the page . . .
(12) Graph \( f(t) = (2-t)(t-4)[u(t-2) - u(t-4)] \).

We do this in two parts. Without the step functions, we would have

\[
f(t) = (2-t)(t-4) = -t^2 + 6t - 8.
\]

This graph is a parabola that crosses the horizontal axis at \( t = 2 \) and \( t = 4 \) and has a maximum turning point at \( (3, 1) \).

The step functions take the following values:

\[
u(t-2) - u(t-4) = \begin{cases} 
0 - 0 = 0, & t < 2 \\
1 - 0 = 1, & 2 < t < 4 \\
1 - 1 = 0, & t > 4 
\end{cases}
\]

When the two parts are multiplied together we get

\[
f(t) = \begin{cases} 
0, & t < 2 \\
\text{parabola}, & 2 < t < 4 \\
0, & t > 4 
\end{cases}
\]

and the graph

We can think of a difference of two step functions

\[u(t - a) - u(t - b)\]

like a mathematical window whose frame obscures the graph of any multiplying function to the sides, i.e. \( t < a \) and \( t > b \).
(13) Determine $f(t)$ from the following graph:

The jumps in the graph can be represented by step functions. The location of the jump is determined by the critical value of the step function, the size of the jump can be achieved by multiplying the step function be a suitably sized constant, the direction of the jump by the sign of the constant. Working from left to right:

- 1ˢᵗ jump: $4u(t - 1)$ up by 4 at $T = 1$
- 2ⁿᵈ jump: $-6u(t - 2)$ down by 6 at $T = 2$
- 3ʳᵈ jump: $4u(t - 3)$ up by 4 at $T = 3$.

Adding these together gives us the function:

$$f(t) = 4u(t - 1) - 6u(t - 2) + 4u(t - 3).$$
(14) Determine $f(t)$ from the following graph:

![Graph](image)

Using basic coordinate geometry, the equation of the oblique straight line is given by

$$y = 3t - 3.$$  

We are only seeing the straight line between $t = 1$ and $t = 2$; this gives a “mathematical window” of

$$[u(t - 1) - u(t - 2)].$$

Multiplying the two parts together gives us our function:

$$f(t) = (3t - 3)[u(t - 1) - u(t - 2)].$$

For more general piece-wise functions we can break them down into fragments, each with their own window, and add the parts together.

**Example**

$$f(t) = \begin{cases} 
   t & , \quad 0 < t < 1 \\
   2 - t & , \quad 1 < t < 2 \\
   0 & , \quad t > 2 
\end{cases}$$

$$f(t) = t[u(t - 0) - u(t - 1)] + (2 - t)[u(t - 1) - u(t - 2)]$$

As far as Laplace transforms are concerned $u(t - 0) \equiv 1$ so this can be re-written as

$$f(t) = t - 2(t - 1)u(t - 1) + (t - 2)u(t - 2).$$
There are two results that we will need in order to solve ODEs containing step functions.

**Result 1**

\[ L[u(t - T)] = \frac{1}{s} e^{-st} \]

**Proof**

\[
L[u(t - T)] = \int_0^\infty u(t - T) e^{-st} \, dt \\
= \int_0^T 0 \, dt + \int_T^\infty 1 \, e^{-st} \, dt \\
= 0 + \left[ -\frac{1}{s} e^{-st} \right]_T^{\infty} \\
= \frac{1}{s} e^{-st} \quad (s > 0).
\]

**Result 2 – The Second Shifting Theorem**

Given a function \( f(t) \) with Laplace transform \( F(s) \), it can be shown that

\[ L[f(t - T) u(t - T)] = F(s) e^{-st}. \]

The proof of this is omitted. The implication of this result is demonstrated graphically below in **Figure 2**:

![Diagram showing an example of the Second Shifting Theorem](image.png)
We shall use modified versions of the following diagram to help us apply the 2nd Shifting Theorem:

\[
\begin{array}{ccc}
  f(t) & \leftrightarrow & F(s) \\
  \downarrow & \updownarrow & \\
  f(t-T)u(t-T) & \leftrightarrow & F(s) e^{-sT}
\end{array}
\]

**Examples**

(16) Invert the Laplace transform \( \frac{1}{s(s + 2)} e^{-4s} \).

For this we are starting at the bottom-right of the diagram and moving round anti-clockwise:

\[
\begin{array}{ccc}
  f(t) & \leftarrow & F(s) \\
  \downarrow & \uparrow & \\
  f(t-T)u(t-T) & F(s) e^{-sT}
\end{array}
\]

**Step 1:** Ignore exponential to give 

\[
F(s) = \frac{1}{s(s + 2)} = \frac{1}{2} \left[ \frac{2}{s(s + 2)} \right].
\]

**Step 2:** Invert \( F(s) \) to give 

\[
f(t) = \frac{1}{2} [1 - e^{-2t}].
\]

**Step 3:** Shift and truncate with \( T = 4 \), i.e. replace \( t \) by \( t - 4 \) and multiply by \( u(t - 4) \), to give the completed inversion

\[
f(t-T)u(t-T) = \frac{1}{2} [1 - e^{-2(t-4)}]u(t-4).
\]
(17) Solve the differential equation \( \ddot{x} + 4x = u(t - 3) \) subject to \( x(0) = \dot{x}(0) = 0 \).

**Transform:** \[
[ s^2 \ddot{x} - s x(0) - x'(0) ] + 4\ddot{x} = \frac{1}{s} e^{-3s}
\]

**Substitute** initial conditions and **solve** for \( \ddot{x} \):

\[
( s^2 + 4) \ddot{x} = \frac{1}{s} e^{-3s}
\]

\[
\ddot{x} = \frac{1}{s(s^2 + 4)} e^{-3s}
\]

**Invert** with the help of the 2\textsuperscript{nd} Shifting Theorem:

Again, we are starting at the bottom-right of the diagram and moving round anti-clockwise:

\[
\begin{array}{ccc}
\hline
f(t) & \leftarrow & F(s) \\
\hline
\downarrow & & \uparrow \\
( t - T) & u(t - T) & F(s) e^{-st} \\
\hline
\end{array}
\]

**Step 1:** Ignore exponential to give \( F(s) = \frac{1}{s(s^2 + 4)} = \frac{1}{4} \left[ \frac{2}{s^2 + 2^2} \right] \).

**Step 2:** Invert \( F(s) \) to give \( f(t) = \frac{1}{4}[1 - \cos(2t)] \).

**Step 3:** Shift and truncate with \( T = 3 \), i.e. replace \( t \) by \( (t - 3) \) and multiply by \( u(t - 3) \), to give \( f(t - T)u(t - T) \) and the completed solution

\[
x(t) = \frac{1}{4}[1 - \cos(2(t - 3))]u(t - 3) .
\]

These two examples illustrate the use of the 2\textsuperscript{nd} Shifting Theorem to invert Laplace transforms that contain an exponential factor. The next example shows the theorem working in the other direction.
(18) Determine \( L[t^2 u(t-10)] \).

This time we go round the diagram the other way:

\[
\begin{array}{c|c}
 f(t) & F(s) \\
\uparrow & \downarrow \\
 f(t-T)u(t-T) & F(s) e^{-st}
\end{array}
\]

We have, however, a slight problem in that the given function of \( t \) does not quite conform to the structure bottom-left. We need to re-write the multiplying function \( t^2 \) so that the variable \( t \) always has a “minus \( T \)”. For this example \( T = 10 \) so we introduce this as follows:

\[
t^2 = [(t-10) + 10]^2 = (t-10)^2 + 20(t-10) + 100.
\]

The original function becomes \( [(t-10)^2 + 20(t-10) + 100]u(t-10) \) which now does conform. Next we use the theorem to transform:

**Step 1:** Cover up the step function and the “minus 10s” to leave

\[
f(t) = t^2 + 20t + 100.
\]

**Step 2:** Transform to give \( F(s) = \frac{2}{s^3} + \frac{20}{s^2} + \frac{100}{s} \).

**Step 3:** Everything we covered up in Step 1 is now accounted for by multiplying \( F(s) \) by \( e^{-10s} \) and so completes the transformation,

\[
L[t^2 u(t-10)] = \left[ \frac{2}{s^3} + \frac{20}{s^2} + \frac{100}{s} \right] e^{-10s}.
\]

Now we shall look at another new type of function, one that is closely related to the step function.
7) The Dirac Delta Function

Consider the graph of a function \( D(t) \) something like the following:

![Graph of \( D(t) \)](Figure 3)

This graph might represent a force applied and then removed, or a voltage switched on and then off. We can then define another function as

\[
U(t) = \int_0^t D(z) \, dz
\]

which records the area under the graph from 0 to the variable \( t \). Assume that the function \( D(t) \) is such that the total area under the graph is 1 unit². This means that

\[
U(t) = \begin{cases} 
0 & , \quad t \leq T - \varepsilon \\
0 \text{ increasing to } 1 & , \quad T - \varepsilon \leq t \leq T + \varepsilon \\
1 & , \quad t \geq T + \varepsilon
\end{cases}
\]

and so the graph of \( U(t) \) will look like

![Graph of \( U(t) \)](Figure 4)
Now suppose that the interval over which $D(t)$ is non-zero is reduced but the height of the graph is increased in such a way that the total area under the graph is still 1 unit$^2$:

![Figure 5](image1.png)

We get what might be termed a "blip". The corresponding graph of $U(t)$ looks like

![Figure 6](image2.png)

Notice that $U(t)$ is beginning to look like the step function. If we let the interval about $T$ shrink further towards zero and the height increase towards infinity (again, in such a way that the area under the graph is always 1 unit$^2$), then, in the limit, we obtain an "instantaneous, infinite blip" at $t = T$. This "infinite blip" at $t = T$ is called the **Dirac delta function** and is denoted by

$$\delta(t - T);$$

it has the graph as shown in **Figure 7** on the next page.
The Dirac delta function has the property that
\[
\int_0^T \delta(z - T) \, dz = \begin{cases} 0 & , \ t < T \\ 1 & , \ t > T \end{cases} = u(t - T) .
\]

Another way of expressing this is that
\[
\delta(t - T) = \frac{d}{dt} u(t - T) .
\]
That is, the Dirac delta function is the derivative of the step function.

The Dirac delta function also has the property that
\[
\int_0^M f(t) \delta(t - T) \, dt = f(T)
\]
for any function \( f \), providing \( M > T \). From this result we can determine the Laplace transform of the Dirac delta function:
\[
L[\delta(t - T)] = \int_0^\infty \delta(t - T) e^{-st} \, dt = e^{-st} .
\]

The Dirac delta function is often used the model actions or events that occur over a short period of time; reality is compressed into an instant of time for mathematical convenience. It is sometimes referred to as the **impulse function**.
Examples

(19) Solve the differential equation \( \dot{x} + 4x = 3 \delta(t - 2) \), \( x(0) = 5 \).

**Transform:** \[
[s \bar{x} - x(0)] + 4\bar{x} = 3e^{-2s}
\]

**Substitute** initial condition and **solve** for \( \bar{x} \):

\[
(s + 4)\bar{x} = 3e^{-2s} + 5
\]

\[
\bar{x} = \frac{3}{(s + 4)}e^{-2s} + \frac{5}{(s + 4)}
\]

**Invert** to give solution:

[A]: \[
\frac{3}{(s + 4)} \xrightarrow{\text{Invert}} 3e^{-4t}
\]

By 2nd Shifting Theorem,

\[
\frac{3}{(s + 4)}e^{-2s} \xrightarrow{\text{Invert}} 3e^{-4(t - 2)}u(t - 2)
\]

[B]: \[
\frac{5}{(s + 4)} \xrightarrow{\text{Invert}} 5e^{-4t}
\]

\[
x(t) = 3e^{-4(t - 2)}u(t - 2) + 5e^{-4t}
\]

(20) In a branch of an electronic circuit, the variation of current with time is modelled by the differential equation

\[
\frac{d^2i}{dt^2} + 36i = \frac{dV}{dt},
\]

where \( V(t) \) is an input voltage. Suppose \( V(t) = 240[u(t - 5) - u(t - 10)] \). Assuming zero initial conditions, determine \( i \) as a function of \( t \).
Referring back to the introduction of the Dirac delta function, we can write
\[ \frac{dV}{dt} = 240[\delta(t - 5) - \delta(t - 10)] \]
and so the differential equation becomes
\[ \frac{d^2i}{dt^2} + 36i = 240[\delta(t - 5) - \delta(t - 10)] . \]

The Laplace transformation process results in
\[ I = \frac{240}{s^2 + 36} \left[ e^{-5s} - e^{-10s} \right] , \]
which can be inverted with the help of the 2\textsuperscript{nd} Shifting Theorem:
\[ i(t) = 40\sin[6(t - 5)] u(t - 5) - 40\sin[6(t - 10)] u(t - 10) . \]

Try and fill in the details for yourselves.

You may recall this warning from earlier in the notes:

\textbf{The Laplace transform of a product is NOT EQUAL TO the product of the individual Laplace transforms.}

We shall now look at a kind of product rule for Laplace transforms.
8) Convolution

The convolution of two functions \( f(t) \) and \( g(t) \) is another function of \( t \) denoted by \( f \ast g \) and defined by

\[
(f \ast g)(t) = \int_0^t f(t - z) g(z) \, dz \quad [a]
\]
or

\[
(f \ast g)(t) = \int_0^t f(z) g(t - z) \, dz . \quad [b]
\]

It can be shown that these two integrals are equal.

If \( F(s) \) and \( G(s) \) are the Laplace transforms of \( f(t) \) and \( g(t) \) respectively, then

\[
L[f \ast g] = F(s) \, G(s) .
\]

Example

(21) Invert the Laplace transform \( \frac{1}{s(s + 4)^2} \).

This could be done using partial fractions (try it!), but convolution could be used as an alternative:

Set \( F(s) = \frac{1}{s} \) and \( G(s) = \frac{1}{(s + 4)^2} \) and invert individually from tables:

\[
f(t) = 1 \quad g(t) = te^{-4t} .
\]

Form the convolution of \( f \) and \( g \) using form \([a] \):

\[
f(t - z) = 1 \quad g(z) = ze^{-4z} ;
\]

\[
f \ast g = \int_0^t f(t - z) g(z) \, dz
\]

\[
= \int_0^t ze^{-4z} \, dz .
\]

Omitting the details, integration by parts gives

\[
f \ast g = -\frac{1}{4}te^{-4t} - \frac{1}{16}e^{-4t} + \frac{1}{16} .
\]
9) A Note on Control Theory

As mentioned earlier, Laplace transforms are an important tool in control theory. Keeping things basic, in a simple control system there may be an input and an output. Control theory is concerned with the relationship between the input and the output within the system. Often a control system can be modelled by a differential equation that relates input to output in what may be referred to as the time domain. For example, a differential equation like

$$\frac{dx}{dt} + kx = v(t)$$

might relate an input \( v(t) \) to an output \( x(t) \). For a given input, the equation is solved to give the corresponding output. However, in control theory, options regarding the input are often left open. Without a specific input, we can’t determine a corresponding output, but valuable information about the systems controllability can be established by leaving the input unspecified, and continuing to apply the solution process to the differential equation in any case.

For the above differential equation, denote the following Laplace transforms:

$$L[x(t)] = X(s) \quad \quad L[v(t)] = V(s).$$

Now take the Laplace transform of the differential equation; assume that the initial condition of \( x(t) \) is zero:

$$s X(s) + kX(s) = V(s).$$

Now re-arrange:

$$(s + k)X(s) = V(s)$$

$$\frac{X(s)}{V(s)} = \frac{1}{s + k}.$$

What we have here is the ratio of the output of the system to its input in what is called the Laplace domain. This ratio is called the system’s transfer function.

In general, for a system with a single input and a single output we have

$$\frac{X(s)}{V(s)} = H(s).$$

The transfer function \( H(s) \) is itself a Laplace transform and its form will depend on the structure of the differential equation modelling the system. The transfer function plays a huge role in control theory as much information can be derived from it. But that is another story for another module!
Tutorial Exercises

(1) With the aid of the table of Laplace transforms, transform the following functions:

(i) $2t - 5$  (ii) $at + b$

(iii) $2t^2 - 3t + 1$  (iv) $(a + bt)^2$

(v) $3te^t$  (vi) $t^2 e^{-2t}$

(vii) $e^{2t-1}$  (viii) $e^{-t} \cos(2t)$

(ix) $\sin(\omega t + \theta)$  (x) $\cos(\omega t + \theta)$

(xi) $\sin^2 t$  (xii) $\cos^2 t$

(xiii) $e^{-2t} \sin(5t)$  (xiv) $e^{-2t} \sin(5t + \frac{\pi}{2})$

(2) Invert each of the following Laplace transforms:

(i) $\frac{5}{s + 3}$  (ii) $\frac{2}{s^2 + 16}$

(iii) $\frac{s + 1}{s^2 + 1}$  (iv) $\frac{s - 2}{s^2 - 4}$

(v) $\frac{s - 4}{s^2 - 4}$  (vi) $\frac{1}{(s + 9)^2}$

(vii) $\frac{s + 2}{(s + 2)^2 + 1}$  (viii) $\frac{s + 4}{(s + 2)^2 + 1}$

(ix) $\frac{1}{s^4}$  (x) $\frac{4}{s^5}$

(xi) $\frac{s}{(s - 1)^2 - 4}$  (xii) $\frac{5s + 1}{(s - 1)(s + 2)}$

(xiii) $\frac{s + 5}{(s + 1)(s - 3)}$  (xiv) $\frac{32}{s(s^2 + 16)}$. 
(3) Using Laplace transforms, solve the following ordinary differential equations:

(i) \( \dot{x} + 2x = 0 \); \( x(0) = 1 \)

(ii) \( \dot{x} + 2x = 1 \); \( x(0) = 0 \)

(iii) \( \dot{x} - 3x = 10 \sin t \); \( x(0) = 0 \)

(iv) \( \ddot{x} - 4x = 0 \); \( x(0) = 0 \), \( \dot{x}(0) = -6 \)

(v) \( \ddot{x} + \omega^2 x = 0 \); \( x(0) = A \), \( \dot{x}(0) = B \)

(vi) \( \ddot{x} + 4\dot{x} + 4x = 0 \); \( x(0) = 2 \), \( \dot{x}(0) = -3 \)

(vii) \( 9\ddot{x} - 6\dot{x} + x = 0 \); \( x(0) = 3 \), \( \dot{x}(0) = 1 \)

(viii) \( \dddot{x} + \dot{x} = 0 \); \( x(0) = 0 \), \( \dot{x}(0) = 4 \)

(4) Sketch the graphs and determine the Laplace transforms of each of the following:

(i) \( 2u(t - 5) \)

(ii) \(-4u(t - 3)\)

(iii) \(3[u(t - 2) - u(t - 4)]\)

(iv) \(u(t - 1) - 2u(t - 2) + u(t - 3)\)

(v) \(u(t - 2)u(t - 4)\)

(vi) \(-2u(t - 3)u(t - 6)\)

(vii) \((t - 3)u(t - 3)\)

(viii) \((t - 5)^2 u(t - 5)\)

(ix) \(tu(t - 3)\)

(x) \(t^2u(t - 5)\)

(xi) \([u(t - 2) - u(t - 4)](t - 2)^2\)

(xii) \([u(t - 1) - u(t - 2)](t - 1)^2 + [u(t - 2) - u(t - 3)](3 - t)\)
(5) Using combinations of step-functions, determine single-line expressions for each of the following functions and (hence) their Laplace transforms:

(i) 

(ii) 

(iii) 

(iv)
(6) Invert each of the following Laplace transforms:

(i) \( \frac{1}{s} e^{-s} \) 
(ii) \( \frac{1}{s} e^{-2s} \) 
(iii) \( \frac{1}{s^2} e^{-s} \) 

(iv) \( \frac{1}{s^2} e^{-2s} \) 
(v) \( \frac{1}{s + 2} e^{-3s} \) 
(vi) \( \frac{s}{s^2 + 9} e^{-5s} \) 

(vii) \( \frac{4}{s^2 + 4} e^{-10s} \) 
(viii) \( \frac{s}{s^2 + 2s + 10} e^{-4s} \) 
(ix) \( \frac{2(e^{-2s} - e^{-4s})}{s^2 + 25} \)
The ODE below models the behaviour of a time-varying current \( i(t) \) in a circuit:

\[
\frac{di}{dt} + i = v(t).
\]

The function on the RHS, \( v(t) \), is a source voltage. Below are various source voltages. For each, graph \( v(t) \) and write as a single-line expression using step-functions. Then determine and graph \( i(t) \).

(i) \( v(t) = \begin{cases} v_0, & 0 \leq t \leq 1 \\ 0, & t \geq 1 \end{cases} \), \( i(0) = v_0 \)

(ii) \( v(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ 0, & t \geq 1 \end{cases} \), \( i(0) = 0 \)

The ODE and initial conditions

\[
m\ddot{x} + c\dot{x} + kx = f(t), \quad x(0) = \dot{x}(0) = 0
\]

models the mass displacement of a spring-mass-dashpot system, where \( m \) is the mass, \( c \) is the damping constant, \( k \) is the spring stiffness and \( f(t) \) is a forcing term. For each of the following forcing terms graph \( f(t) \) and express it as a single-line expression. Setting \( m = 1, c = 2 \) and \( k = 5 \), determine \( x \) in terms of \( t \) (i.e. solve the ODE).

(i) \( f(t) = \begin{cases} 1, & 0 \leq t \leq 10 \\ 0, & t \geq 10 \end{cases} \)

(ii) \( f(t) = \begin{cases} t, & 0 \leq t \leq 10 \\ 0, & t \geq 10 \end{cases} \)

Evaluate the following integrals:

(i) \( \int_{0}^{2} \delta(t - 1) \, dt \) \hspace{1cm} (ii) \( \int_{0}^{2} \delta(t - 3) \, dt \) \hspace{1cm} (iii) \( \int_{2}^{4} \delta(t - 1) \, dt \)

(iv) \( \int_{0}^{3} t \delta(t - 2) \, dt \) \hspace{1cm} (v) \( \int_{0}^{5} e^{-t} \delta(t - 3) \, dt \) \hspace{1cm} (vi) \( \int_{0}^{2} e^{-t} \delta(t - 3) \, dt \)
(10) Write down the Laplace transforms of

(i) \(2\, \delta(t - 1)\)  
(ii) \(4\, \delta(t - 2)\)  
(iii) \(6\, \delta(t)\)

(11) Invert the following Laplace transforms:

(i) \(\frac{s + 1}{s}\)  
(ii) \(\frac{s + 1}{s} e^{-2s}\)  
(iii) \(\frac{s + 2}{s + 1} e^{-4s}\)  
(iv) \(\frac{s^2 + 2s}{s^2 + 9} e^{-5s}\)

(12) Solve the following ODEs:

(i) \(\ddot{x} + 4x = 2\, \delta(t)\), \(x(0) = 0\)
(ii) \(\ddot{x} + 4x = 2\, \delta(t - 1)\), \(x(0) = 0\)
(iii) \(\ddot{x} + 4x = -4\, \delta(t - \pi)\), \(x(0) = 0\), \(\dot{x}(0) = 4\)
(iv) \(\dddot{x} + 4\dot{x} + 5x = \delta(t - 1)\), \(x(0) = \dot{x}(0) = 0\)

(13) The ODE and initial condition

\[
\frac{di}{dt} + i = \dot{v}(t) \quad , \quad i(0) = 0
\]

models the behaviour of a time-varying current in a circuit. The function \(\dot{v}(t)\) is a source voltage. For each of the source voltages below, write down \(\dot{v}(t)\); hence determine and graph \(i(t)\):

(i) \(v(t) = v_0 \, u(t - 1)\)
(ii) \(v(t) = v_0 \, [u(t - 1) - u(t - 2)]\) .
Answers

(1) (i) \( \frac{2}{s^2} - \frac{5}{s} \)

(ii) \( \frac{a}{s^2} + \frac{b}{s} \)

(iii) \( \frac{4}{s^3} - \frac{3}{s^2} + \frac{1}{s} \)

(iv) \( \frac{a^2}{s} + \frac{2ab}{s^2} + \frac{2b^2}{s^3} \)

(v) \( \frac{3}{(s - 1)^2} \)

(vi) \( \frac{2}{(s + 2)^3} \)

(vii) \( \frac{e^{-1}}{s - 2} \)

(viii) \( \frac{s + 1}{(s + 1)^2 + 4} \)

(ix) \( \frac{s \sin \theta + \omega \cos \theta}{s^2 + \omega^2} \)

(x) \( \frac{s \cos \theta - \omega \sin \theta}{s^2 + \omega^2} \)

(xi) \( \frac{2}{s( s^2 + 4) } \)

(xii) \( \frac{s^2 + 2}{s( s^2 + 4) } \)

(xiii) \( \frac{5}{(s + 2)^2 + 25} \)

(xiv) \( \frac{(s + 2) \sin \frac{\omega}{4} + 5 \cos \frac{\omega}{4} }{(s + 2)^2 + 25} \)

(2) (i) \( 5e^{-3t} \)

(ii) \( \frac{1}{2} \sin (4t) \)

(iii) \( \cos t + \sin t \)

(iv) \( e^{-2t} \) or \( \cosh(2t) - \sinh(2t) \)

(v) \( \frac{3}{2} e^{-2t} - \frac{1}{2} e^{2t} \)

(vi) \( t e^{-9t} \)

or

\( \cosh(2t) - 2 \sinh(2t) \)

(vii) \( e^{-2t} \cos t \)

(viii) \( e^{-2t} \) (\( \cos t + 2 \sin t \))

(ix) \( \frac{1}{6} t^3 \)

(x) \( \frac{1}{6} t^4 \)

(xi) \( e^t \left[ \cosh(2t) + \frac{1}{2} \sinh(2t) \right] \)

(xii) \( 2e^t + 3e^{-2t} \)

(xiii) \( 2e^{3t} - e^{-t} \)

(xiv) \( 2 \left[ 1 - \cos(4t) \right] \)
(3) (i) \( x = e^{-2t} \) \hspace{1cm} \text{(ii)} \( x = \frac{1}{2} (1 - e^{-2t}) \)

(iii) \( x = e^{3t} - \cos t - 3 \sin t \) \hspace{1cm} \text{(iv)} \( x = -3 \sinh(2t) \)

or \( x = -\frac{3}{2} (e^{2t} - e^{-2t}) \)

(v) \( x = A \cos(\omega t) + \frac{B}{\omega} \sin(\omega t) \) \hspace{1cm} \text{(vi)} \( x = (t + 2) e^{-2t} \)

(vii) \( x = 3 e^{t/3} \) \hspace{1cm} \text{(viii)} \( x = 4 (1 - e^{-t}) \)

(4) (i) \( \frac{2}{s} e^{-5s} \) \hspace{1cm} \text{(ii)} \( -\frac{4}{s} e^{-3s} \)

(iii) \( \frac{3}{s} (e^{-2s} - e^{-4s}) \) \hspace{1cm} \text{(iv)} \( \frac{1}{s} (e^{-s} - 2 e^{-2s} + e^{-3s}) \)

(v) \( \frac{1}{s} e^{-4s} \) \hspace{1cm} \text{(vi)} \( -\frac{2}{s} e^{-6s} \)

(vii) \( \frac{1}{s^2} e^{-3s} \) \hspace{1cm} \text{(viii)} \( \frac{2}{s^2} e^{-5s} \)

(ix) \( \left( \frac{1}{s^2} + \frac{3}{s} \right) e^{-3s} \) \hspace{1cm} \text{(x)} \( \left( \frac{2}{s^3} + \frac{10}{s^2} + \frac{25}{s} \right) e^{-5s} \)

(xi) \( \frac{2}{s^3} e^{-2s} - \left( \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s} \right) e^{-4s} \)

(xii) \( \frac{2}{s^3} e^{-s} - \left( \frac{2}{s^3} + \frac{3}{s^2} \right) e^{-2s} + \frac{1}{s^2} e^{-3s} \)
(5) (i) \[5 \left[ u(t - 2) - u(t - 4) \right]\]
\[\frac{5}{s} \left( e^{-2s} - e^{-4s} \right)\]

(ii) \[u(t - 2) + 2u(t - 3) - 3u(t - 4)\]
\[\frac{1}{s} \left( e^{-2s} + 2e^{-3s} - 3e^{-4s} \right)\]

(iii) \[3u(t - 2) - 2u(t - 3) - u(t - 4)\]
\[\frac{1}{s} \left( 3e^{-2s} - 2e^{-3s} - e^{-4s} \right)\]

(iv) \[(t - 5) \left[ u(t - 5) - u(t - 10) \right]\]
\[\frac{1}{s^2} e^{-5s} - \left( \frac{1}{s} + \frac{5}{s} \right) e^{-10s}\]

(v) \[t \left[ 1 - u(t - 5) \right]\]
\[\frac{1}{s^2} - \left( \frac{1}{s} + \frac{5}{s} \right) e^{-5s}\]

(vi) \[(t - 5)u(t - 5) - 2(t - 10)u(t - 10) + (t - 15)u(t - 15)\]
\[\frac{1}{s^2} \left( e^{-5s} - 2e^{-10s} + e^{-15s} \right)\]

(vii) \[(5 - t) \left[ 1 - u(t - 5) \right]\]
\[\left( \frac{5}{s} - \frac{1}{s^2} \right) + \frac{1}{s^2} e^{-5s}\]
\[(6) \quad \begin{align*}
(i) & \quad u(t - 1) \\
(ii) & \quad u(t - 2) \\
(iii) & \quad (t - 1) u(t - 1) \\
(iv) & \quad (t - 2) u(t - 2) \\
(v) & \quad e^{-2(t - 3)} u(t - 3) \\
(vi) & \quad \cos[3(t - 5)] u(t - 5) \\
(vii) & \quad 2 \sin[2(t - 10)] u(t - 10) \\
(vii) & \quad e^{-(t-4)} \left[ \cos[3(t - 4)] - \frac{1}{3} \sin[3(t - 4)] \right] u(t - 4) \\
(viii) & \quad \frac{2}{5} \left\{ \sin[5(t - 2)] u(t - 2) - \sin[5(t - 4)] u(t - 4) \right\}
\end{align*}\]

\[(7) \quad \begin{align*}
(i) & \quad i(t) = v_o - v_o [1 - e^{-(t-1)}] u(t - 1) \\
(ii) & \quad i(t) = (e^{-t} + t - 1) - (t - 1) u(t - 1)
\end{align*}\]

\[(8) \quad \begin{align*}
(i) & \quad x(t) = \frac{1}{5} \left[ f_1(t) - f_1(t - 10) u(t - 10) \right]
\quad \text{where} \\
& \quad f_1(t) = 1 - e^{-t} \cos(2t) - \frac{1}{2} e^{-t} \sin(2t) \\
(ii) & \quad x(t) = g_1(t) - g_1(t - 10) u(t - 10) - 2 g_2(t - 10) u(t - 10)
\quad \text{where} \\
& \quad g_1(t) = \frac{1}{2} \left[ 5t - 2 + 2 e^{-t} \cos(2t) - \frac{3}{2} e^{-t} \sin(2t) \right] \\
& \quad g_2(t) = 1 - e^{-t} \cos(2t) - \frac{1}{2} e^{-t} \sin(2t) 
\end{align*}\]
(9) (i) 1 (ii) 0 (iii) 0
(iv) 4 (v) $e^{-3}$ (vi) 0

(10) (i) $2 e^{-x}$ (ii) $4 e^{-2x}$ (iii) 6

(11) (i) $\delta(t) + 1$
(ii) $\delta(t - 2) + u(t - 2)$
(iii) $\delta(t - 4) + e^{-(t - 4)} u(t - 4)$
(iv) $\delta(t - 5) + \{2 \cos[3(t - 5)] - 3 \sin[3(t - 5)]\} u(t - 5)$

(12) (i) $x(t) = 2 e^{-4t}$
(ii) $x(t) = 2 e^{-4(t - 1)} u(t - 1)$
(iii) $x(t) = 2 \sin(2t) [1 - u(t - \pi)]$
(iv) $x(t) = e^{-2(t - 1)} \sin(t - 1) u(t - 1)$

(13) (i) $i(t) = v_0 e^{-(t - 1)} u(t - 1)$
(ii) $i(t) = v_0 \left[ e^{-(t - 1)} u(t - 1) - e^{-(t - 2)} u(t - 2) \right]$
TABLE OF LAPLACE TRANSFORMS

$L[f(t)]$ is defined by $\int_0^\infty f(t) e^{-st} \, dt$

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(s) = L[f(t)]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{s}$</td>
</tr>
<tr>
<td>$t$</td>
<td>$\frac{1}{s^2}$</td>
</tr>
<tr>
<td>$t^2$</td>
<td>$\frac{2}{s^3}$</td>
</tr>
<tr>
<td>$t^{n-1}$</td>
<td>$\frac{(n-1)!}{s^n}$</td>
</tr>
<tr>
<td>$e^{-\alpha t}$</td>
<td>$\frac{1}{(s + \alpha)}$</td>
</tr>
<tr>
<td>$te^{-\alpha t}$</td>
<td>$\frac{1}{(s + \alpha)^2}$</td>
</tr>
<tr>
<td>$1 - e^{-\alpha t}$</td>
<td>$\frac{\alpha}{s(s + \alpha)}$</td>
</tr>
<tr>
<td>$\sin(\omega t)$</td>
<td>$\frac{\omega}{s^2 + \omega^2}$</td>
</tr>
<tr>
<td>$\cos(\omega t)$</td>
<td>$\frac{s}{s^2 + \omega^2}$</td>
</tr>
<tr>
<td>$1 - \cos(\omega t)$</td>
<td>$\frac{\omega^2}{s(s^2 + \omega^2)}$</td>
</tr>
<tr>
<td>$\omega t \sin(\omega t)$</td>
<td>$\frac{2\omega^2 s}{(s^2 + \omega^2)^2}$</td>
</tr>
</tbody>
</table>

Over . . .
<table>
<thead>
<tr>
<th>Expression</th>
<th>Transformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin(\omega t) - \omega t \cos(\omega t) )</td>
<td>( \frac{2 \omega^3}{(s^2 + \omega^2)^2} )</td>
</tr>
<tr>
<td>( \sin(\omega t + \phi) )</td>
<td>( \frac{s \sin \phi + \omega \cos \phi}{s^2 + \omega^2} )</td>
</tr>
<tr>
<td>( e^{-\alpha t} \sin(\omega t) )</td>
<td>( \frac{\omega}{(s + \alpha)^2 + \omega^2} )</td>
</tr>
<tr>
<td>( e^{-\alpha t} \cos(\omega t) )</td>
<td>( \frac{s + \alpha}{(s + \alpha)^2 + \omega^2} )</td>
</tr>
<tr>
<td>( e^{-\alpha t} \left( \cos(\omega t) - \frac{\alpha}{\omega} \sin(\omega t) \right) )</td>
<td>( \frac{s}{(s + \alpha)^2 + \omega^2} )</td>
</tr>
<tr>
<td>( e^{-\alpha t} + \frac{\alpha}{\omega} \sin(\omega t) - \cos(\omega t) )</td>
<td>( \frac{\alpha^2 + \omega^2}{(s + \alpha)(s^2 + \omega^2)} )</td>
</tr>
<tr>
<td>( \sinh(\beta t) )</td>
<td>( \frac{\beta}{s^2 - \beta^2} )</td>
</tr>
<tr>
<td>( e^{-\alpha t} \sinh(\beta t) )</td>
<td>( \frac{\beta}{(s + \alpha)^2 - \beta^2} )</td>
</tr>
<tr>
<td>( \cosh(\beta t) )</td>
<td>( \frac{s}{s^2 - \beta^2} )</td>
</tr>
<tr>
<td>( e^{-\alpha t} \cosh(\beta t) )</td>
<td>( \frac{s + \alpha}{(s + \alpha)^2 - \beta^2} )</td>
</tr>
<tr>
<td>( e^{-\alpha t} f(t) )</td>
<td>( F(s + \alpha) )</td>
</tr>
<tr>
<td>( x )</td>
<td>( \bar{x} )</td>
</tr>
<tr>
<td>( \dot{x} )</td>
<td>( s \bar{x} - x(0) )</td>
</tr>
<tr>
<td>( \ddot{x} )</td>
<td>( s^2 \bar{x} - s x(0) - \dot{x}(0) )</td>
</tr>
<tr>
<td><strong>Unit Step Function</strong></td>
<td></td>
</tr>
<tr>
<td>------------------------</td>
<td>------------------------</td>
</tr>
<tr>
<td>$u(t - T) = \begin{cases} 0 &amp; , \ t &lt; T \ 1 &amp; , \ t &gt; T \end{cases}$</td>
<td>$\frac{1}{s} e^{-sT}$</td>
</tr>
</tbody>
</table>

| $f(t - T) u(t - T)$ | $F(s) e^{-sT}$ |

<table>
<thead>
<tr>
<th><strong>Dirac Delta Function</strong></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta(t)$</td>
<td>1</td>
</tr>
</tbody>
</table>

| $\delta(t - T)$ | $e^{-sT}$ |

<table>
<thead>
<tr>
<th><strong>Convolution Integral</strong></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\int_{0}^{t} f(z) g(t - z) , dz$</td>
<td>$F(s) G(s)$</td>
</tr>
</tbody>
</table>

OR

$\int_{0}^{t} f(t - z) g(z) \, dz$ |