School of Computing, Engineering & Built Environment

Mathematics Summer School

Level 2 Entry – Engineering

Integral Calculus

Contents

Integral Calculus

1). Antidifferentiation or Indefinite Integration

Definition: A function $F(x)$ is called an antiderivative of $f(x)$ if

$$F'(x) = f(x) \quad .$$

The problem we shall now look at is how to determine $F(x)$ from $f(x)$.

Consider the following. Which function when differentiated gives $2x$?

If we recall the rules of differentiation it should be obvious that

$$\frac{d}{dx}\left[x^2 \right] = 2x \quad .$$

Therefore x^2 is an antiderivative of $2x$. We say “an” antiderivative because there are others:

$$\frac{d}{dx}\left[x^2 + 3 \right] = 2x$$

$$\frac{d}{dx}\left[x^2 - 13 \right] = 2x$$

$$\frac{d}{dx}\left[x^2 + \pi \right] = 2x \quad .$$

In fact, any function of the form

$$F(x) = x^2 + C \quad ,$$

where C is a constant, is an antiderivative.

Further Example

(1). $f(x) = x^3$ $\qquad$ $F(x) = \frac{1}{4}x^4 + C$

The process of determining antiderivatives is usually referred to as **integration**. We use the following notation:

$$\underbrace{\int}_{\text{Integral sign}} \underbrace{f(x)}_{\text{Integrand}}\, \underbrace{dx}_{\text{Differential}} = \underbrace{F(x)}_{\text{Antiderivative}} + \underbrace{C}_{\text{Constant of integration}}$$

Notes: **(i).** The differential dx indicates the **variable of integration**. We could be integrating with respect to another variable, say t, in which case we would have

$$\int f(t)\, dt = F(t) + C \ .$$

(ii). Because of the unspecified constant C, we call this process **indefinite** integration.

Example

(2). Determine $\int 5x^4 \, dx$.

$$I = \int 5x^4 \, dx$$

$$= \underline{\underline{x^5 + C}} \ .$$

2). The Rules of Integration and the Integrals of Some Basic Functions

(a). $$\int k\, f(x)\, dx = k \int f(x)\, dx \qquad \text{(where } k \text{ is a constant)}$$

(b). $$\int [\, f(x) + g(x)\,]\, dx = \int f(x)\, dx + \int g(x)\, dx$$

(c). $$\int k\, dx = k\, x + C \qquad \text{(where } k \text{ is a constant)}$$

(d). $$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C \qquad \underline{\underline{(\, n \neq -1\,)}} \ .$$

3). The Integrals of Some Other Functions

$f(x)$	$F(x) + C$
$\cos(x)$	$\sin(x) + C$
$\sin(x)$	$-\cos(x) + C$
e^x	$e^x + C$
$\dfrac{1}{x}$	$\ln\lvert x\rvert + C$

Recall from Differential Calculus that all "trig angles" must be in **radians** and not in degrees.

We shall regard the above as our **table of standard integrals**. It will be advantageous to be able to think of this table in terms of other variables (e.g. t, u, etc.) as well as x.

Using the **rules** and the **table** we can integrate simple functions.

Examples

(3). Determine $I = \int (5x - 2)\, dx$.

$$I = \int (5x - 2)\, dx$$

$$= 5\int x\, dx - \int 2\, dx$$

$$= 5\,[\tfrac{1}{2}x^2] - 2x + C$$

$$= \underline{\underline{\tfrac{5}{2}x^2 - 2x + C}}$$

(4). Determine $I = \int \sqrt{x}\ dx$.

First write the square root as a power.

$$I = \int x^{1/2}\ dx$$

$$= \frac{x^{3/2}}{3/2} + C$$

$$= \underline{\underline{\tfrac{2}{3}x^{3/2} + C}}$$

(5). Determine $I = \int \frac{1 + x}{x^3}\ dx$.

Note that we **cannot** integrate the top and bottom of the fraction separately. In order to integrate, we divide out:

$$I = \int \frac{1 + x}{x^3}\ dx$$

$$= \int \left[\frac{1}{x^3} + \frac{x}{x^3} \right] dx$$

$$= \int \left[\frac{1}{x^3} + \frac{1}{x^2} \right] dx$$

$$= \int [\ x^{-3} + x^{-2}\ ]\ dx$$

$$= \frac{x^{-2}}{-2} + \frac{x^{-1}}{-1} + C$$

$$= -\tfrac{1}{2}x^{-2} - x^{-1} + C$$

$$= \underline{\underline{-\frac{1}{2x^2} - \frac{1}{x} + C}}$$

(6). Determine $I = \int \frac{1 + x}{x^2}\, dx$.

This is similar to the previous example, but with a subtle variation in the denominator.

$$I = \int \frac{1 + x}{x^2}\, dx$$

$$= \int \left[\frac{1}{x^2} + \frac{x}{x^2} \right] dx$$

$$= \int \left[\frac{1}{x^2} + \frac{1}{x} \right] dx$$

$$= \int [\, x^{-2} + x^{-1} \,]\, dx$$

Note the power of -1 in the second term

$$= \frac{x^{-1}}{-1} + \ln|x| + C$$

$$= -x^{-1} + \ln|x| + C$$

$$= -\frac{1}{x} + \ln|x| + C$$

Note the natural log!

Sometimes we will have additional information that will allow us to determine a specific value for the constant of integration, *C*.

Examples

(7). Determine $I = \int (4x - x^2)\, dx$ if $I = 21$ when $x = 3$.

$$I = \int (4x - x^2)\, dx$$

$$I = 2x^2 - \tfrac{1}{3}x^3 + C\ .$$

Set $I = 21$ and $x = 3$, then solve for C:

$$21 = 2 \times 3^2 - \tfrac{1}{3} \times 3^3 + C \quad \rightarrow \quad 21 = 18 - 9 + C \quad \rightarrow \quad C = 12$$

$$I = 2x^2 - \tfrac{1}{3}x^3 + 12\,.$$

(8). A projectile is travelling in a straight line at a constant velocity u. At time $t = 0$ the projectile starts accelerating at a ms^{-2}. Determine expressions for the projectile's subsequent velocity and displacement.

Let the velocity be denoted by v:

$$\frac{dv}{dt} = a$$

Integrate to eliminate the derivative:

$$v = \int a\, dt \qquad \text{(Note that } a \text{ is a constant.)}$$

$$v = at + C_1 .$$

When $t = 0$, $v = u$:

$$u = a \times 0 + C_1 \qquad \rightarrow \qquad C_1 = u$$

$$v = at + u$$

or

$$\underline{\underline{v = u + at}} .$$

Now determine displacement. Let the displacement be denoted by s:

$$\frac{ds}{dt} = v$$

$$s = \int v\, dt \qquad \text{(Note that } v \text{ is } \textbf{not} \text{ a constant.)}$$

$$s = \int (u + at)\, dt$$

$$s = ut + \tfrac{1}{2}at^2 + C_2 .$$

Assume that $s = 0$ when $t = 0$:

$$0 = u \times 0 + \tfrac{1}{2}a \times 0^2 + C_2 \qquad \rightarrow \qquad C_2 = 0$$

and so

$$\underline{\underline{s = ut + \tfrac{1}{2}at^2}} .$$

4). Indefinite Integration by Substitution

From the table of standard integrals we see that

$$\int \cos(x)\,dx \;=\; \sin(x) \;+\; C \quad ,$$

but what is $\int \cos(2x)\,dx$?

The answer is **NOT** simply $\sin(2x) \;+\; C$.

Our table is our **standard** set. If we have an integral that is **non-standard**, that is a variant of a standard integral or something more complicated, we may be able to convert it into a standard form by a substitution or change of variable.

Consider

$$I \;=\; \int \cos(2x)\,dx \quad .$$

To make the integrand standard, we need a variable in the cosine with a coefficient of 1. To get this, set

$$u \;=\; 2x \quad . \qquad [1]$$

This turns $\cos(2x)$ into $\cos(u)$. However the differential must now reflect the change of variable, so we must express dx in terms of du. We do this by differentiating [1],

$$\frac{du}{dx} \;=\; 2 \quad ,$$

then splitting this derivative term into its differentials to give

$$du \;=\; 2\,dx$$

or

$$dx \;=\; \tfrac{1}{2}du \quad . \qquad [2]$$

[1] and [2] together turn the non-standard integral into

$$I \;=\; \int \cos(u)\left(\tfrac{1}{2}du\right)$$

$$=\; \tfrac{1}{2}\int \cos(u)\,du \qquad \text{(standard in terms of } u\text{)}$$

$$=\; \tfrac{1}{2}\sin(u) \;+\; C \quad .$$

Substituting back for u in terms of the original variable x gives

$$\underline{\underline{I \;=\; \tfrac{1}{2}\sin(2x) \;+\; C}} \quad .$$

Further Examples

(9). (i). Determine $I = \int \cos(4x)\, dx$.

$$I = \int \cos(4x)\, dx$$

Set: $u = 4x$

Differentiate: $\dfrac{du}{dx} = 4$

$du = 4\,dx$

$dx = \tfrac{1}{4}\,du$

$$I = \int \cos(u)\,(\tfrac{1}{4}du)$$

$$= \tfrac{1}{4}\int \cos(u)\, du$$

$$= \tfrac{1}{4}\sin(u) + C$$

$$= \underline{\underline{\tfrac{1}{4}\sin(4x) + C}}$$

(9). (ii). By using the substitution $u = ax + b$ and following the above pattern we can show that

$$I = \int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C\,,$$

(providing that a and b are constants).

Note: $\int \cos(ax+b)\,dx$ is called a **linear variant** of $\int \cos(x)\,dx$.

(10). (i). Determine $I = \int \sin(4x+3)\,dx$

$$I = \int \sin(4x+3)\,dx$$

Set: $u = 4x+3$

Differentiate: $\dfrac{du}{dx} = 4$

$du = 4\,dx$

$dx = \tfrac{1}{4}du$

$$I = \int \sin(u)\,(\tfrac{1}{4}du)$$

$$= \tfrac{1}{4}\int \sin(u)\,du$$

$$= -\tfrac{1}{4}\cos(u) + C$$

$$= \underline{\underline{-\tfrac{1}{4}\cos(4x+3) + C}}$$

(10). (ii). By using the substitution $u = ax+b$ we can generalise the above result for any linear variant of the integral of $\sin(x)$:

$$I = \int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C,$$

(providing that a and b are constants).

(11). (i). Determine $I = \int e^{-t}\, dt$.

$$I = \int e^{-t}\, dt$$

Set: $u = -t$

Differentiate: $\dfrac{du}{dt} = -1$

$$du = -dt$$

$$dt = -du$$

$$I = \int e^{u}\,(-du)$$

$$= -\int e^{u}\, du$$

$$= -e^{u} + C$$

$$= \underline{\underline{-e^{-t} + C}}$$

(11). (ii). General result:

$$I = \int e^{x}\, dx = e^{x} + C$$

$$I = \int e^{ax+b}\, dx = \frac{1}{a} e^{ax+b} + C \ .$$

(12). (i). Determine $I = \int \frac{dx}{2x-4}$.

$$I = \int \frac{dx}{2x-4}$$

Set: $u = 2x - 4$

Differentiate: $\frac{du}{dx} = 2$

$$du = 2\,dx$$

$$dx = \tfrac{1}{2}\,du$$

$$I = \int \frac{1}{u}\left(\tfrac{1}{2}du\right)$$

$$= \tfrac{1}{2}\int \frac{1}{u}\,du$$

$$= \tfrac{1}{2}\ln|u| + C$$

$$= \underline{\underline{\tfrac{1}{2}\ln|2x-4| + C}}$$

(12). (ii). General result:

$$I = \int \frac{1}{x}\,dx = \ln|x| + C$$

$$I = \int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C \ .$$

Summarising the general results derived by the substitution method we have

$f(x)$	$F(x) + C$
$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b) + C$
$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b) + C$
e^{ax+b}	$\frac{1}{a}e^{ax+b} + C$
$\frac{1}{ax+b}$	$\frac{1}{a}\ln\lvert ax+b\rvert + C$

providing a and b are constants.

The substitution technique can also be used with integrals of the form

$$I = \int [g(x)]^n h(x)\, dx$$

providing

$$h(x) = a\frac{d}{dx}[g(x)]$$

where a is some multiplying constant.

Examples

(13). $$I = \int 6x(3x^2+7)^5\, dx$$

$$= \int (3x^2+7)^5(6x\,dx)$$

continued over . . .

Set: $u = 3x^2 + 7$,

Differentiate:

$$\frac{du}{dx} = 6x$$

$$du = 6x\,dx \ .$$

$$I = \int u^5 \ du$$

$$= \tfrac{1}{6}u^6 + C$$

$$= \tfrac{1}{6}(3x^2 + 7)^6 + C \ .$$

(14). Determine $I = \int x e^{-x^2} \ dx$.

$$I = \int e^{-x^2} x \ dx$$

Set: $u = -x^2$

Differentiate:

$$\frac{du}{dx} = -2x$$

$$du = -2x\,dx$$

$$x\,dx = -\tfrac{1}{2}du$$

$$I = \int e^u (-\tfrac{1}{2}du)$$

$$= -\tfrac{1}{2}\int e^u \ du$$

$$= -\tfrac{1}{2} e^u + C$$

$$= \underline{\underline{-\tfrac{1}{2} e^{-x^2} + C}}$$

(15). (i). Now consider integrals of the form

$$I = \int \frac{f'(x)}{f(x)}\,dx$$

where the integrand is a fraction whose numerator (top bit) is the derivative of its denominator (bottom bit).

If we set $u = f(x)$, then differentiate and separate to give $du = f'(x)\,dx$, the integral can be recast as

$$I = \int \frac{du}{u} = \int \frac{1}{u}\,du = \ln|u| + C = \ln|f(x)| + C \ .$$

That is,

$$\underline{\underline{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C}} \ .$$

You may use this as a general result.

(15). (ii). Determine $I = \displaystyle\int \frac{x}{x^2 + 4}\,dx$.

If we "fine tune" this integral by re-writing it as

$$I = \tfrac{1}{2}\int \frac{2x}{x^2 + 4}\,dx \ ,$$

the numerator is now the exact derivative of the denominator and so we can use the result above:

$$I = \tfrac{1}{2}\int \frac{2x}{x^2 + 4}\,dx$$

$$= \underline{\underline{\tfrac{1}{2}\ln|x^2 + 4| + C}} \ .$$

(16). Sometimes it is not obvious that a change of variable will help in the integration process. Consider

$$I = \int x \sqrt{x-2}\ dx \quad .$$

Set: $\quad u = x - 2$

Differentiate: $\quad \dfrac{du}{dx} = 1$

$du = dx$

This means that the integral can be written as

$$I = \int x \sqrt{u}\ du \quad .$$

To complete the change we must write x in terms of u : $\quad u = x - 2$

$x = u + 2$

$$\begin{aligned} I &= \int (u + 2) \sqrt{u}\ du \\ &= \int (u + 2)\, u^{1/2}\ du \\ &= \int (u^{3/2} + 2u^{1/2})\ du \\ &= \tfrac{2}{5} u^{5/2} + \tfrac{4}{3} u^{3/2} + C \\ &= \underline{\underline{\tfrac{2}{5} (x-2)^{5/2} + \tfrac{4}{3} (x-2)^{3/2} + C}} \end{aligned}$$

Integration by substitution is an extremely important technique. Spotting whether a substitution will work, and what that substitution is, can be tricky but will become easier, the more problems you tackle.

5). Integration by Parts

Another useful technique for integrating 'difficult' functions is **integration by parts**. The method is derived from the product rule for differentiation which states that if f and g are both functions of x then :

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)] \ .$$

The rule can be written in a more compact form as

$$\frac{d}{dx}[f\,g] = f\,g' + f'g\,.$$

Rearranging this expression yields

$$f\,g' = \frac{d}{dx}[f\,g] - f'g\,.$$

Integrating both sides, with respect to x, gives the rule for integration by parts,

$$\int f\,g'dx = f\,g - \int f'g\,dx$$

which enables us to integrate many functions involving products of functions of x.

The goal here is to obtain a less complicated integral on the right-hand-side, i.e. $\int f'g\,dx$ than the one we started with, i.e. $\int f\,g'dx$.

Application of the method requires us to identify candidates for f, the function to be differentiated, and g', the function to be integrated.

Examples

(17). (i). Determine $\int x\sin(x)\,dx$ using integration by parts.

Write the integral as

$$I = \int x\sin(x)dx$$

and set

$$f(x) = x \qquad\qquad g'(x) = \sin(x) \ .$$

Note that we normally select $f(x)$ as the part of the product that results in a 'simpler' function when differentiated.

Differentiating $f(x)$ and integrating $g'(x)$ gives:

$$f'(x) = 1 \qquad\qquad g(x) = -\cos(x) \ .$$

Applying the rule for integration by parts:

$$\begin{aligned} I &= \int f\, g'\, dx = f\ g - \int f' g\ dx \\ &= -x\cos(x) - \int 1\left(-\cos(x)\right)dx \\ &= -x\cos(x) + \int \cos(x)\ dx \ . \end{aligned}$$

The remaining integral is one of our standard integrals and can easily be determined:

$$\begin{aligned} I &= -x\cos(x) + \int \cos(x)\ dx \\ &= \underline{\underline{-x\cos(x) + \sin(x) + C}} \ . \end{aligned}$$

The required constant of integration is introduced at the end.

(17). (ii). Determine $\int x^2 \sin(x)\ dx$ using integration by parts.

Following the pattern of part (a), write the integral as:

$$I = \int x^2 \sin(x)\ dx$$

and set

$$f(x) = x^2 \qquad\qquad g'(x) = \sin(x) \ .$$

Once again we select $f(x)$ as the part of the product that simplifies when differentiated. Now integrate $g'(x)$ and differentiate $f(x)$:

$$f'(x) = 2x \qquad\qquad g(x) = -\cos(x) \ .$$

Applying the formula:

$$\begin{aligned} I &= \int f\, g'\, dx = f\, g - \int f' g\, dx \\ &= -x^2 \cos(x) - \int 2x[-\cos(x)]\, dx \\ &= -x^2 \cos(x) + 2\int x\cos(x)\, dx \qquad (*) \end{aligned}$$

The remaining integral is still non-standard and requires a second integration by parts:

$$f(x) = x \qquad\qquad g'(x) = \cos(x) \ .$$

$$f'(x) = 1 \qquad\qquad g(x) = \sin(x) \ .$$

Applying the formula:

$$\begin{aligned} \int x\cos(x)\, dx &= x\sin(x) - \int 1.\sin(x)\, dx \\ &= x\sin(x) - \int \sin(x)\, dx \\ &= x\sin(x) + \cos(x) \end{aligned}$$

Substituting this result back into (*) above and adding a '+ C' gives

$$\begin{aligned} I &= -x^2 \cos(x) + 2\int \cos(x)\, x\, dx \\ &= -x^2 \cos(x) + 2\,[x\sin(x) + \cos(x)] + C \\ &= \underline{\underline{-x^2 \cos(x) + 2\,x\sin(x) + 2\cos(x) + C}} \end{aligned}$$

(18). Now consider $I = \int x^3 \ln x\, dx$.

Sometimes we have to think a little harder as to which part of the product should be chosen as $f(x)$ and which part should be chosen as $g'(x)$. In this example we do not know how to integrate $\ln x$ but we do know how to differentiate it. We therefore choose to differentiate $\ln x$ and integrate the x^3 term. Define

$$f(x) = \ln x \qquad\qquad g'(x) = x^3 \ ,$$

giving

$$f'(x) = \frac{1}{x} \qquad\qquad g(x) = \tfrac{1}{4}x^4 \ .$$

Substituting the appropriate expressions into the integration by parts formula gives

$$\begin{aligned} I &= f\ g \quad - \int f' g\, dx \\ &= \ln x \,.\, \tfrac{1}{4}x^4 \quad - \int \frac{1}{x} . \tfrac{1}{4}x^4\, dx \\ &= \tfrac{1}{4}x^4 \ln x \quad - \tfrac{1}{4}\int x^3\, dx \\ &= \tfrac{1}{4}x^4 \ln x \quad - \tfrac{1}{4}\tfrac{1}{4}\, x^4 \ + \ C \\ &= \underline{\underline{\tfrac{1}{4}x^4 \ln x \quad - \tfrac{1}{16}\, x^4 \ + \ C}} \end{aligned}$$

(19). Now consider $I \ = \displaystyle\int e^{3x} \sin(2x)\; dx$.

Note: This example is messy, but it does illustrate a "nice" application of repeated integration by parts.

If we perform a repeated integration by parts, each time setting f as the trigonometric function and g' as the exponential function, we obtain

$$I \ = \ \tfrac{1}{3}e^{3x}\sin(2x) \ - \ \tfrac{2}{9}e^{3x}\cos(2x) \ - \ \tfrac{4}{9}\int e^{3x}\sin(2x)\; dx \quad .$$

The original integral has re-appeared in the last term on the right-hand-side and so it looks like the approach is futile. However, we can replace the integral by the letter I, because that is how we defined it at the start:

$$I \ = \ \tfrac{1}{3}e^{3x}\sin(2x) \ - \ \tfrac{2}{9}e^{3x}\cos(2x) \ - \ \tfrac{4}{9}I \ .$$

This is now an equation for I. Since I is what we are after, solving this equation should give us the integral (providing we remember it must contain a $+$ C):

$$\begin{aligned} I \ + \ \tfrac{4}{9}I \ &= \ \tfrac{1}{3}e^{3x}\sin(2x) \ - \ \tfrac{2}{9}e^{3x}\cos(2x) \\ \tfrac{13}{9}I \ &= \ \tfrac{1}{3}e^{3x}\sin(2x) \ - \ \tfrac{2}{9}e^{3x}\cos(2x) \\ I \ &= \ \tfrac{9}{13}\left[\tfrac{1}{3}e^{3x}\sin(2x) \ - \ \tfrac{2}{9}e^{3x}\cos(2x)\right] \ + \ C \\ I \ &= \ \tfrac{1}{13}\left[3e^{3x}\sin(2x) \ - \ 2e^{3x}\cos(2x)\right] \ + \ C \\ \underline{\underline{I \ }} &\underline{\underline{= \ \tfrac{1}{13}\left[3\sin(2x) \ - \ 2\cos(2x)\right]e^{3x} \ + \ C}} \quad . \end{aligned}$$

6). Integration Using Trig Identities

Integrals that contain powers of sines or cosines are non-standard. In such cases the following trigonometric identities may prove useful:

$$\sin^2(\omega x) = \tfrac{1}{2}[1 - \cos(2\omega x)]$$

$$\cos^2(\omega x) = \tfrac{1}{2}[1 + \cos(2\omega x)] \ .$$

Examples

(20). Determine $I = \displaystyle\int \sin^2(2x)\,dx$.

$$\begin{aligned} I &= \int \sin^2(2x)\,dx \\ &= \int \tfrac{1}{2}[1 - \cos(4x)]\,dx \\ &= \tfrac{1}{2}[x - \tfrac{1}{4}\sin(4x)] + C \\ &= \underline{\underline{\tfrac{1}{2}x - \tfrac{1}{8}\sin(4x) + C}} \ . \end{aligned}$$

(21). Determine $I = \displaystyle\int \cos^2(3x)\,dx$.

$$\begin{aligned} I &= \int \cos^2(3x)\,dx \\ &= \int \tfrac{1}{2}[1 + \cos(6x)]\,dx \\ &= \tfrac{1}{2}[x + \tfrac{1}{6}\sin(6x)] + C \\ &= \underline{\underline{\tfrac{1}{2}x + \tfrac{1}{12}\sin(6x) + C}} \ . \end{aligned}$$

7). Integration Using Partial Fractions

Partial fractions can sometimes be used to re-write an integrand in a more friendly form.

Example

(22). Determine $I = \int \frac{2x + 5}{(2x + 1)(x + 3)}\, dx$ with the aid of partial fractions.

Applying a partial fraction expansion, it can be shown that:

$$\frac{2x + 5}{(2x + 1)(x + 3)} = \frac{8/5}{2x + 1} + \frac{1/5}{x + 3} .$$

This means that the integral can be re-written as

$$I = \tfrac{8}{5} \int \frac{1}{(2x + 1)}\, dx + \tfrac{1}{5} \int \frac{1}{(x + 3)}\, dx .$$

Each of the sub-integrals can be determined from the result:

$$\int \frac{1}{ax + b}\, dx = \frac{1}{a} \ln|ax + b| + C .$$

So

$$I = \tfrac{8}{5} \int \frac{1}{(2x + 1)}\, dx + \tfrac{1}{5} \int \frac{1}{(x + 3)}\, dx$$

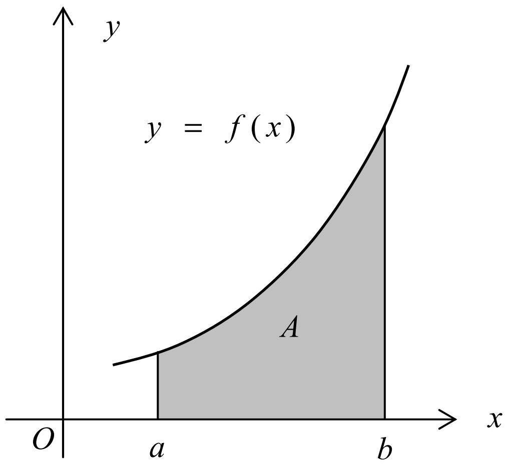
$$= \tfrac{8}{5}\, \tfrac{1}{2} \ln|2x + 1| + \tfrac{1}{5} \ln|x + 3| + C$$

$$= \underline{\underline{\tfrac{4}{5} \ln|2x + 1| + \tfrac{1}{5} \ln|x + 3| + C}} .$$

8). Definite Integration

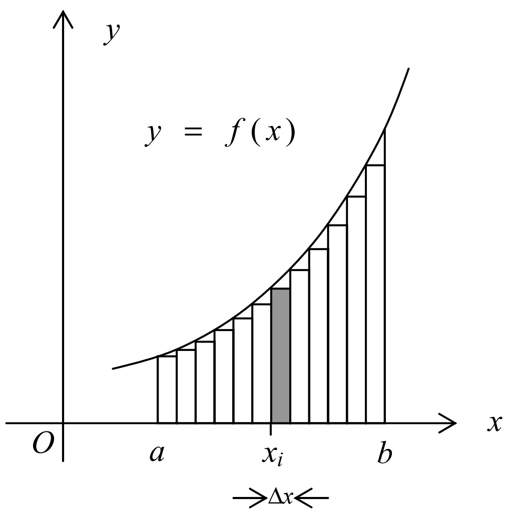
(a). Definition and Evaluation of a Definite Integral

Consider the graph of a function $y = f(x)$ and denote the function's integral by $F(x) + C$:



Integration can allow us to work out the area A of the shaded region between the curve $y = f(x)$, the x-axis and the vertical lines $x = a$ and $x = b$.

Figure 1a



Approximate the area using n narrow rectangles.

Area of a typical rectangle:

$$f(x_i)\,\Delta x \ .$$

Approximation of total area:

$$A \approx A_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$

Figure 1b

By squeezing more and more rectangles into the area, we obtain better and better approximations to A. It can be shown that

$$A_n \;\to\; A = \int_a^b f(x)\,dx = \big[\,F(x)\,\big]_a^b = F(b) - F(a) \ .$$

This is a statement of the Fundamental Theorem of Integral Calculus.

The integral

$$\int_a^b f(x)\, dx$$

is called a **definite integral**, with a and b the **limits of integration**.

Notes: **(i).** Sometimes the integral is written as

$$\int_{x=a}^{x=b} f(x)\, dx \quad .$$

If you always keep in mind the variable of integration (x in this case) you may drop the "$x =$" 's.

(ii). Although definite integration is introduced through area under a curve, it does have far wider interpretations and applications. We shall use the area interpretation later. In the meantime, just learn how to evaluate definite integrals.

Example

(23). Evaluate $\int_0^1 x^2\, dx$.

$$\begin{aligned} I &= \int_0^1 x^2\, dx \\ &= \left[\, \tfrac{1}{3} x^3 \,\right]_0^1 \\ &= \tfrac{1}{3}\, 1^3 - \tfrac{1}{3}\, 0^3 \\ &= \underline{\underline{\tfrac{1}{3}}} \;. \end{aligned}$$

Further Examples

(24). Evaluate $\int_{-1}^{2} (x^2 - 1)\, dx$.

$$I = \int_{-1}^{2} (x^2 - 1)\, dx$$

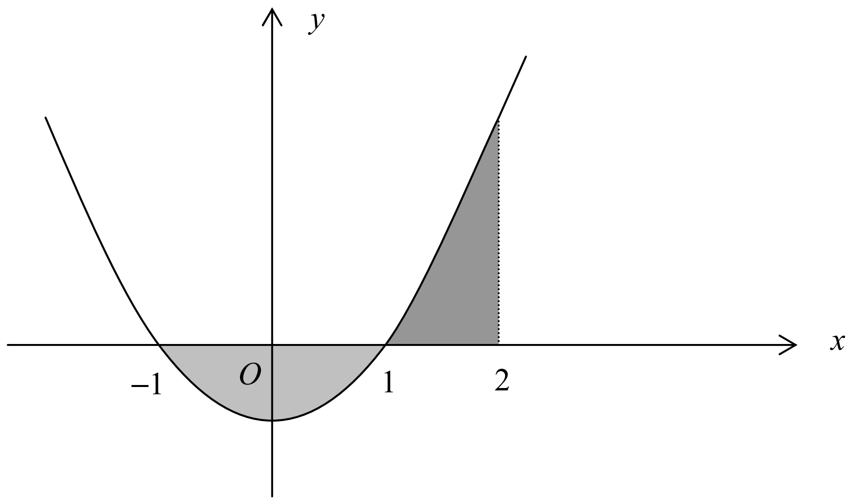
$$= \left[\tfrac{1}{3}x^3 - x \right]_{-1}^{2}$$

$$= \left(\tfrac{1}{3}\times 2^3 - 2 \right) - \left(\tfrac{1}{3}\times(-1)^3 - (-1) \right)$$

$$= \left(\tfrac{8}{3} - 2 \right) - \left(-\tfrac{1}{3} + 1 \right)$$

$$= \tfrac{8}{3} - 2 + \tfrac{1}{3} - 1$$

$$= \underline{\underline{0}}$$



Area below the x-axis is negative and cancels out the positive area above the x-axis.

(25). Evaluate $\int_0^{\pi/4} \sin(2x)\ dx$.

$$I = \int_0^{\pi/4} \sin(2x)\ dx$$

$$= \left[-\tfrac{1}{2}\cos(2x)\right]_0^{\pi/4}$$

$$= -\tfrac{1}{2}\left[\cos(\tfrac{\pi}{2}) \ - \ \cos(0)\right]$$

Remember: Radians, not degrees!

$$= -\tfrac{1}{2}[\ \ 0 \ \ - \ \ 1\ \ ]$$

$$= \underline{\underline{\tfrac{1}{2}}}$$

(b). Effect of a Change of Variable (Integration by Substitution)

If we transform an integral using a substitution, we can incorporate the limits in one of two ways. We can treat the integral as indefinite and only evaluate with the limits after the integration has been done and the resulting expression has been transformed back to the original variable of integration. Alternatively, we can transform the limits as we go along.

Example

(26). Evaluate $\int_0^1 (2x \ - \ 1)^2\ dx$

Method 1 is to treat the integral initially as indefinite and leave the evaluation at the limits until the end:

$$I = \int_0^1 (2x \ - \ 1)^2\ dx$$

$$u = 2x - 1$$

$$\frac{du}{dx} = 2$$

$$du = 2\,dx$$

$$dx = \tfrac{1}{2}\,du$$

$$I = \int_0^1 (2x - 1)^2 \, dx$$

$$= \int u^2 \left(\tfrac{1}{2}\, du\right) \qquad \text{(temporarily omit the limits)}$$

$$= \tfrac{1}{2} \int u^2 \, du$$

$$= \tfrac{1}{6} u^3 \qquad \text{(can omit the + C)}$$

$$= \tfrac{1}{6} (2x - 1)^3 \qquad \text{(now reintroduce limits)}$$

$$= \left[\tfrac{1}{6} (2x - 1)^3\right]_0^1$$

$$= \tfrac{1}{6} 1^3 - \tfrac{1}{6} (-1)^3$$

$$= \tfrac{1}{6} + \tfrac{1}{6}$$

$$= \underline{\underline{\tfrac{1}{3}}}$$

Method 2 is to transform the limits as we go:

$$I = \int_{x=0}^{x=1} (2x - 1)^2 \, dx$$

$$u = 2x - 1$$

$$\frac{du}{dx} = 2$$

$$du = 2\,dx$$

$$dx = \tfrac{1}{2}\, du$$

Limits:

$$x = 0 \rightarrow u = -1$$

$$x = 1 \rightarrow u = +1$$

$$I = \tfrac{1}{2}\int_{u=-1}^{u=+1} u^2 \, du$$

$$= \left[\tfrac{1}{6} u^3 \right]_{u=-1}^{u=+1}$$

$$= \tfrac{1}{6}(+1)^3 - \tfrac{1}{6}(-1)^3$$

$$= \tfrac{1}{6} + \tfrac{1}{6}$$

$$= \underline{\underline{\tfrac{1}{3}}}$$

Notes: **(i).** A result that sometimes proves useful is

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx \quad .$$

(ii). Definite integration has a few restrictions. One problem arises when the integrand and its antiderivative are undefined at or between the limits of integration. If this is so we may not be able to evaluate the integral. For example,

$$\int_{-1}^{0} \frac{1}{x^2}\,dx \quad , \quad \int_{0}^{1} \frac{1}{x^2}\,dx \quad \text{and} \quad \int_{-1}^{1} \frac{1}{x^2}\,dx$$

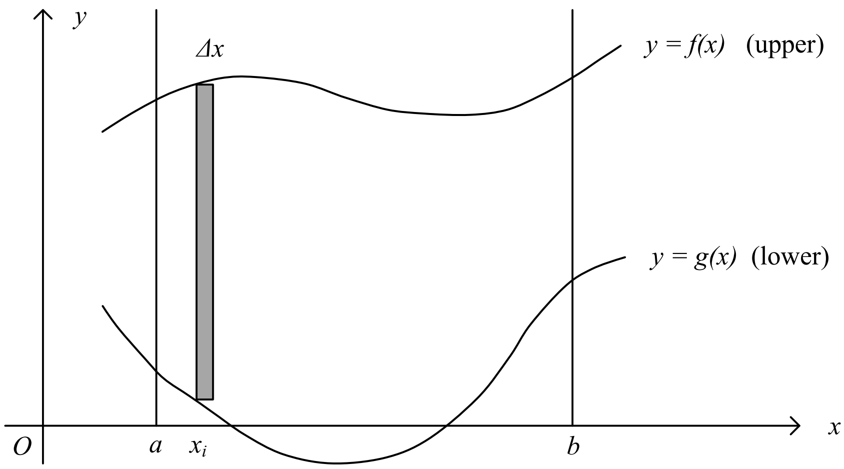
all cannot be evaluated because of problems at $x = 0$. However,

$$\int_{-2}^{-1} \frac{1}{x^2}\,dx \quad \text{and} \quad \int_{1}^{2} \frac{1}{x^2}\,dx$$

can be evaluated since the problem value is outside the limits.

(c). Using Definite Integration to Determine Areas Between Curves

Consider the area enclosed between the curves $y = f(x)$ and $y = g(x)$, restricted by the vertical lines $x = a$ and $x = b$:



The area of the rectangular strip located at $x = x_i$ is given by

$$\Delta A = [f(x_i) - g(x_i)]\,\Delta x \ .$$

The total area is approximated by summing over all strips covering the area:

$$A \approx \sum_{i=1}^{n} [f(x_i) - g(x_i)]\,\Delta x \ .$$

The exact area is given by the limit of this expression as $n \to \infty$ and $\Delta x \to 0$, which is the definite integral

$$A = \int_a^b [f(x) - g(x)]\,dx \ .$$

Note that we don't have to establish whether or not the curves lie wholly or partly above or below the x-axis; the formula automatically takes this into account. However a sketch of the curves may be useful to establish which is upper and which is lower. Also, a little bit of analysis may be necessary to determine the limits of the integration.

Example

(27). Determine the area in the 1st quadrant enclosed by the curves $y = x$ and $y = x^3$.

First determine where the two curves intersect by equating the y's, rearranging and solving for x:

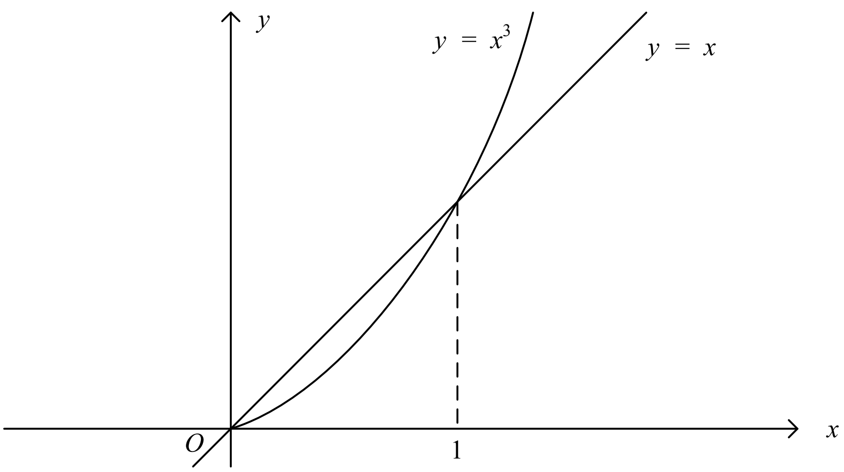
$$x^3 = x$$

$$x^3 - x = 0$$

$$x(x^2 - 1) = 0$$

$$x(x + 1)(x - 1) = 0 \quad .$$

Points of intersection are located at $x = -1$, $x = 0$ and $x = +1$, giving the graphs in the 1st quadrant as:



The enclosed area is therefore given by the definite integral

$$A = \int_0^1 (x - x^3)\, dx$$

$$= \left[\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right]_0^1$$

$$= \left[\tfrac{1}{2} - \tfrac{1}{4}\right] - 0$$

$$= \tfrac{1}{4}$$

Tutorial Exercises

Basic Integration

Q1. Determine the following indefinite integrals:

(i). $\int x^6 \, dx$

(ii). $\int x^7 \, dx$

(iii). $\int (x^3 + 4) \, dx$

(iv). $\int (x^5 + 4x^3 - 7x^2) \, dx$

(v). $\int (1 - 2y + 7y^4) \, dy$

(vi). $\int (7 - 4y + 11y^3) \, dy$

(vii). $\int (t^3 + 3t^4) \, dt$

(viii). $\int (t^{3/2} + 3t - 7) \, dt$

(ix). $\int \frac{x^3 + 1}{x^3} \, dx$

(x). $\int \frac{x^2 + x + 1}{x^2} \, dx$

(xi). $\int x^2 \sqrt{x} \, dx$

(xii). $\int \left(1 + \frac{3}{\sqrt{x}} \right) dx$

(xiii). $\int (1 + \sin x) \, dx$

(xiv). $\int (2 + \cos x) \, dx$.

Integration by Substitution

Q2. Using an appropriate substitution, determine the following indefinite integrals:

(i). $\int 3x^2 (x^3 - 4)^5 \, dx$

(ii). $\int 10x^4 (2x^5 - 4)^3 \, dx$

(iii). $\int 6x(3x^2 + 1)^4 \, dx$

(iv). $\int x^4 (3x^5 - 8)^6 \, dx$

(v). $\int 3x^5 (5x^6 - 10)^4 \, dx$

(vi). $\int (3x^2 + 7)(x^3 + 7x)^8 \, dx$

(vii). $\int (3x^2 + 7)(x^3 + 7x + 1)^8 \, dx$

(viii). $\int \frac{2x - 6}{(x^2 - 6x)^4} \, dx$

(ix). $\int \frac{x^2 + 1}{(x^3 + 3x + 6)} \, dx$

(x). $\int \frac{x + 1}{(2x^2 + 4x - 5)} \, dx$.

Q3. Using an appropriate substitution, determine the following indefinite integrals:

(i). $\int \sin(3x)\ dx$

(ii). $\int \cos(2x-1)\ dx$

(iii). $\int e^{2x}\ dx$

(iv). $\int e^{-x}\ dx$

(v). $\int e^{3x-1}\ dx$

(vi). $\int x e^{x^2}\ dx$

(vii). $\int x \sin(3x^2+1)\ dx$

(viii). $\int x^2 \cos(x^3+1)\ dx$.

Integration by Parts

Q4. Determine the following indefinite integrals using integration by parts:

(i). $\int x e^{x}\ dx$

(ii). $\int x e^{2x}\ dx$

(iii). $\int x e^{-x}\ dx$

(iv). $\int x \sin(4x)\ dx$

(v). $\int x \sin(3x)\ dx$

(vi). $\int x \cos(2x)\ dx$

(vii). $\int x^2 e^{x}\ dx$

(viii). $\int x^2 \cos(2x)\ dx$

(ix). $\int x^2 \ln x\ dx$

(x). $\int x \ln x\ dx$

Q5. Determine

$$\int \ln(x)\ dx$$

by writing it as

$$\int 1 \,.\, \ln(x)\ dx$$

and using integration by parts where

$$f(x) = \ln(x) \quad \text{and} \quad g'(x) = 1 \ .$$

Integration by Re-writing of the Integrand

Q6. Determine the following definite integrals:

(i). $\int \sin^2(4x)\,dx$

(ii). $\int \cos^2(5x)\,dx$

(iii). $\int \dfrac{3x+5}{(x+1)(x+3)}\,dx$

(iv). $\int \dfrac{2x-1}{(x-2)(x+4)}\,dx$

(v). $\int \dfrac{5x^2+15x-2}{(x-1)(x+2)^2}\,dx$.

Definite Integration

Q7. Evaluate the following definite integrals:

(i). $\int_1^2 2x\,dx$

(ii). $\int_1^2 (2x+1)\,dx$

(iii). $\int_0^3 18x^8\,dx$

(iv). $\int_{-1}^1 (3x^2+1)\,dx$

(v). $\int_{-1}^4 x\sqrt{3x^2+1}\,dx$

(vi). $\int_0^1 e^{2x}\,dx$

(vii). $\int_0^3 e^{-x}\,dx$

(viii). $\int_0^1 \sin x\,dx$

(ix). $\int_0^{\pi/2} \cos(2x)\,dx$

(x). $\int_1^2 \dfrac{1}{x}\,dx$

(xi). $\int_0^2 \dfrac{1}{x+1}\,dx$

(xii). $\int_{-\pi/8}^{+\pi/8} \cos(4x)\,dx$.

Q8. Evaluate the following definite integrals with the aid of a change of variable:

(i). $\displaystyle\int_1^2 \frac{x}{1+x^2}\,dx$

(ii). $\displaystyle\int_1^2 \frac{x}{\sqrt{1+x^2}}\,dx$

(iii). $\displaystyle\int_0^1 \frac{x^2}{(1+3x^3)^4}\,dx$

(iv). $\displaystyle\int_0^4 x\sqrt{9+x^2}\,dx$.

Areas by Integration

Q9. Determine the areas under the following curves for the ranges indicated:

(i). $y = x^3$ from $x = 1$ to $x = 3$

(ii). $y = 4 - x^2$ from $x = 0$ to $x = 2$.

Q10. (i). Sketch the curve $y = x(x-1)(x-2)$ showing where it cuts the x-axis.

(ii). Determine the area enclosed by the curve $y = x(x-1)(x-2)$ and the x-axis.

Q11. In the following, determine where the curves intersect, sketch the curves and determine the area enclosed between the curves:

(i). $y = x + 1$ and $y = x^2 - 1$

(ii). $y = 4x$ and $y = x(10 - x)$

(iii). $y = x^2 + 3$ and $y = 2x^2 - 1$.

Answers to Tutorial Exercises

A1. **(i).** $\frac{1}{7}x^7 + C$

(ii). $\frac{1}{8}x^8 + C$

(iii). $\frac{1}{4}x^4 + 4x + C$

(iv). $\frac{1}{6}x^6 + x^4 - \frac{7}{3}x^3 + C$

(v). $y - y^2 + \frac{7}{5}y^5 + C$

(vi). $7y - 2y^2 + \frac{11}{4}y^4 + C$

(vii). $\frac{1}{4}t^4 + \frac{3}{5}t^5 + C$

(viii). $\frac{2}{5}t^{5/2} + \frac{3}{2}t^2 - 7t + C$

(ix). $x - \dfrac{1}{2x^2} + C$

(x). $x + \ln x - \dfrac{1}{x} + C$

(xi). $\frac{2}{7}x^{7/2} + C$

(xii). $x + 6x^{1/2} + C$

(xiii). $x - \cos x + C$

(xiv). $2x + \sin x + C$.

A2. **(i).** $\frac{1}{6}(x^3 - 4)^6 + C$

(ii). $\frac{1}{4}(2x^5 - 4)^4 + C$

(iii). $\frac{1}{5}(3x^2 + 1)^5 + C$

(iv). $\frac{1}{105}(3x^5 - 8)^7 + C$

(v). $\frac{1}{50}(5x^6 - 10)^5 + C$

(vi). $\frac{1}{9}(x^3 + 7x)^9 + C$

(vii). $\frac{1}{9}(x^3 + 7x + 1)^9 + C$

(viii). $-\dfrac{1}{3(x^2 - 6x)^3} + C$

(ix). $\frac{1}{3}\ln\left|x^3 + 3x + 6\right| + C$

(x). $\frac{1}{4}\ln\left|2x^2 + 4x - 5\right| + C$.

A3. **(i).** $-\frac{1}{3}\cos(3x) + C$

(ii). $\frac{1}{2}\sin(2x - 1) + C$

(iii). $\frac{1}{2}e^{2x} + C$

(iv). $-e^{-x} + C$

(v). $\frac{1}{3}e^{3x-1} + C$

(vi). $\frac{1}{2}e^{x^2} + C$

(vii). $-\frac{1}{6}\cos(3x^2 + 1) + C$

(viii). $\frac{1}{3}\sin(x^3 + 1) + C$.

A4. **(i).** $(x-1)e^{x}+C$

(ii). $\frac{1}{4}(2x-1)\,e^{2x}+C$

(iii). $-(x+1)\,e^{-x}+C$

(iv). $\frac{1}{16}[\sin(4x)-4x\cos(4x)]+C$

(v). $\frac{1}{9}[\sin(3x)-3x\cos(3x)]+C$

(vi). $\frac{1}{4}[\cos(2x)+2x\sin(2x)]+C$

(vii). $(x^2-2x+2)\,e^{x}+C$

(vii). $\frac{1}{2}x\cos(2x)+\frac{1}{4}(2x^2-1)\sin(2x)+C$

(ix). $\frac{1}{3}x^3\ \ln|x| - \frac{1}{9}x^3 + C$

(x). $\frac{1}{2}x^2\ \ln|x| - \frac{1}{4}x^2 + C$

A5. $x\ \ln|x| - x + C$

A6. **(i).** $\frac{1}{2}[x - \frac{1}{8}\sin(8x)] + C$

(ii). $\frac{1}{2}[x + \frac{1}{10}\sin(10x)] + C$

(iii). $\ln|x+1| + 2\ln|x+3| + C$ or $\ln\left|(x+1)(x+3)^2\right| + C$

(iv). $\frac{1}{2}\ln|x-2| + \frac{3}{2}\ln|x+4| + C$ or $\ln\left|(x-2)^{1/2}\ (x+4)^{3/2}\right| + C$

(v). Partial fractions: $$\frac{5x^2+15x-2}{(x-1)(x+2)^2} = \frac{2}{x-1} + \frac{3}{x+2} + \frac{4}{(x+2)^2}$$

Integral: $2\ln|x-1| + 3\ln|x+2| - \dfrac{4}{x+2} + C$

A7. **(i).** 3 **(ii).** 4

(iii). 39366 **(iv).** 4

(v). $\frac{335}{9}$ **(vi).** $\frac{1}{2}(e^2 - 1) \approx 3.195$

(vii). $1 - e^{-3} \approx 0.950$ **(viii).** 0.4597

(ix). 0 **(x).** $\ln(2) \approx 0.693$

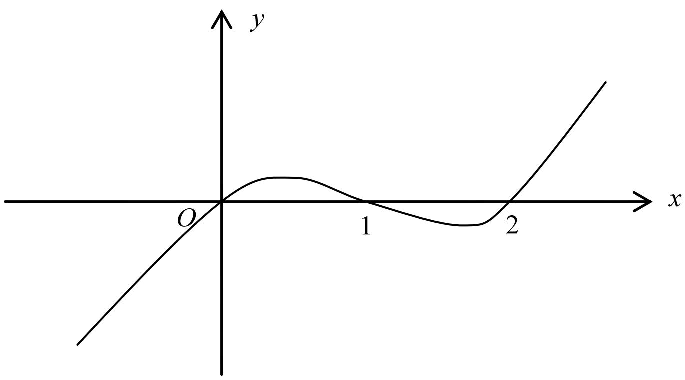
(xi). $\ln(3) \approx 1.099$ **(xii).** 0.5 .

A8. **(i).** $\frac{1}{2}\ln(\frac{5}{2}) \approx 0.4581$ **(ii).** $\sqrt{5} - \sqrt{2} \approx 0.8219$

(iii). $\frac{7}{192} \approx 0.03646$ **(iv).** $\frac{98}{3} \approx 32.6667$

A9. **(i).** 20 **(ii).** $\frac{16}{3}$

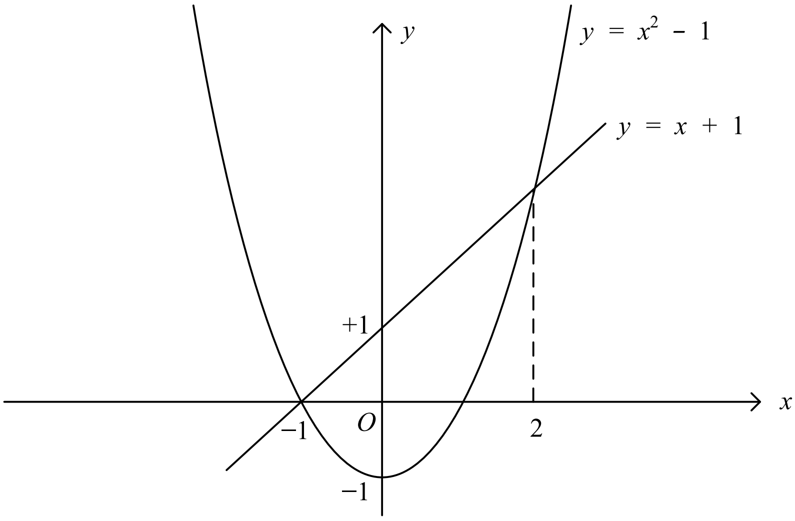
A10. **(i).**



(ii). Integrate between 0 and 1 to give 0.25.
Integrate between 1 and 2 to give −0.25.

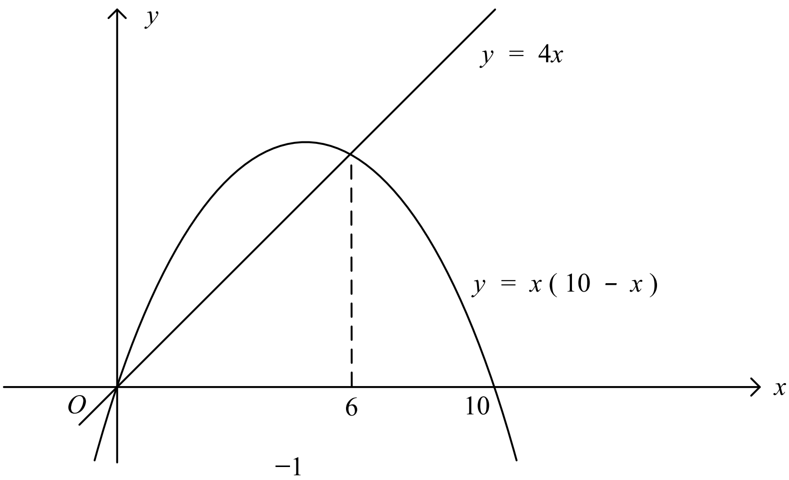
Total area enclosed = 0.25 + 0.25 = 0.5 .

A11. (i).



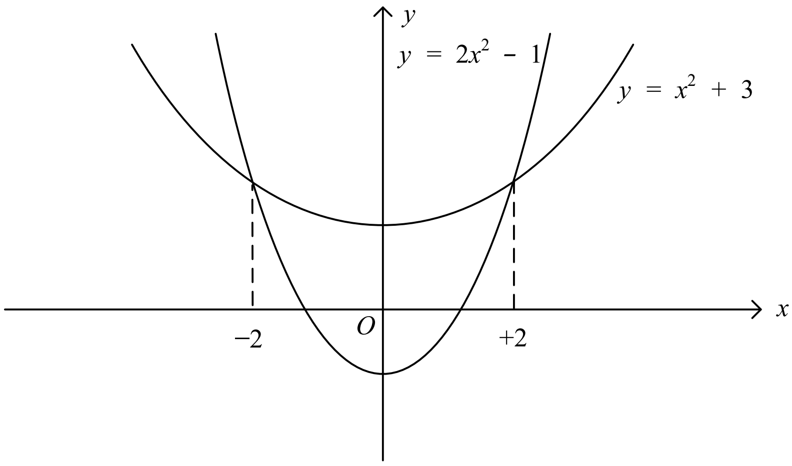
Enclosed area $= \int_{-1}^{2} (x - x^2 + 2)\, dx = \frac{9}{2} = 4.5$

(ii).



Enclosed area $= \int_{0}^{6} (6x - x^2)\, dx = 36$

(iii).



Enclosed area $= \int_{-2}^{+2} (4x - x^2)\,dx = \frac{32}{3}$