School of Computing, Engineering & Built Environment

Mathematics Summer School

Level 3 Entry – Computing

Numbers & Number Systems

Dr Calum A. Macdonald

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1. Introduction

The unit starts with a general introduction on some basic properties of integers and we define widely used terms such as **multiple** and **factor** before presenting the **division algorithm**. We then look at **prime numbers** and **prime factorisation** and describe methods for calculating quantities such as the **greatest common divisor** and **least common multiple** of two integers. A brief discussion on **modular arithmetic** follows to include a look at modulo 2 matrix arithmetic. The unit closes with a discussion on **number systems**, focussing on the three main systems that occur in computing applications; decimal (base 10), binary (base 2) and hexadecimal (base 16). We describe how numbers are represented in each system and present methods for converting between the three bases. A (very) brief discussion is also presented on conversions involving other bases such as octal (base 8).

2. Numbers and Their Properties

This section provides some basic definitions and properties of the integers, i.e. the set

$$\mathbb{Z} = \{ 0, \ \pm 1, \ \pm 2, \ . \ . \ . \},$$

and the related set of natural numbers (positive integers)

$$\mathbb{N} = \{ 1, \ 2, \ 3, \ . \ . \ . \}.$$

2.1. Multiples and Factors

Let m and n be integers. If $n = km$, for some integer k, then n is called a **multiple** of m.

Another way to express the relationship between m and n is to say that m is a **factor** of n. A **factor** of a given number is a number that divides **exactly** into that number. Alternatively we say that n is **divisible** by m, or m **divides** n and write $m \mid n$. If m does not divide n we write $m \nmid n$.

Example 1

(i).   60 is a multiple of 3 since $60 = 3 \times 20$

(i).   36 is a multiple of 9 since $36 = 9 \times 4$

(iii).   22 is not a multiple of 3 as there is no integer which multiplied by 3 gives 22.

Example 2

(i).   The numbers 1, 2, 3, 4, 6 and 12 are all factors of 12 as they each divide exactly into 12.

(ii).   The numbers 1, 2, 4, 5, 10 and 20 are all factors of 20 as they each divide exactly into 20.

(iii). The number 3 is not a factor of 20 as 3 does not divide exactly into 20.

2.2. The Division Algorithm

Let a and b be integers with $b > 0$. Then there exist unique integers q and r such that

$$a = bq + r \text{ with } 0 \leq r < b.$$

The numbers q and r are called the **quotient** and **remainder** respectively when a is divided by b.

Example 3

(i). Let $a = 34$ and $b = 6$. Dividing a by b gives that $q = 5$ and $r = 4$. We can easily check this is correct as, $34 = 6 \times 5 + 4$.

(ii). Let $a = -34$ and $b = 6$. When dividing a by b it might be tempting to write that $q = -5$ and $r = -4$. However, this would be incorrect as r must be non-negative. The correct answer is, $q = -6$ and $r = 2$. We can easily check this is correct as $-34 = 6 \times (-6) + 2$.

2.3. Prime and Composite Numbers

A **prime number** is a positive integer, greater than 1, that is divisible only by 1 and itself.

A **composite number** is a positive integer greater than 1 which is not prime, i.e. it has at least one factor other than 1 and itself.

Example 4

(i). 7 is a prime number, since it is only divisible by 1 and 7.
(ii). 13 is a prime number, since it is only divisible by 1 and 13.
(iii). 4 is a composite (not prime) number, since it is divisible by 1, 2 and 4.
(iii). 12 is a composite (not prime) number, since it is divisible by 1, 2, 3, 4 and 6.

The prime numbers less than 25 are 2, 3, 5, 7, 11, 13, 17, 19, 23.

The composite numbers less than 25 are: 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24.

Notes

(a). The numbers 0 and 1 are neither prime nor composite.
(b). The only even prime number is 2.
(c). In order to determine whether a number, n, is prime it is sufficient to show that it is not divisible by any of the integers from 2 up to the square root of n.
(d). There are infinitely many primes!

2.3.1. Prime Factors
The **prime factors** of a number are the prime numbers that divide that number exactly with no remainder.

Example 5
Find all prime factors of 40.

Solution
The factors of 40 are 1, 2, 4, 5, 8, 10, 20, 40. Only 2 and 5 are prime numbers and so the prime factors of 40 are 2 and 5.

2.3.2. Prime Factorisation
The **fundamental theorem of arithmetic** states that every positive integer greater than 1 is either prime or can be written as a product of prime numbers in a unique way (except for the order of the factors). The process is known as **prime factorisation**.

Example 6
(i).   Find the prime factorisation of 368.

(ii).   Find the prime factorisation of 8085.

(ii).   Find the prime factorisation of 2381400.

Solution:
(i).   First note that 368 is an even number and can therefore be divided by 2, i.e.

$$368 \quad = \quad 2 \times 184$$
$$= 2 \times 2 \times 92$$
$$= 2 \times 2 \times 2 \times 46$$
$$= 2 \times 2 \times 2 \times 2 \times 23 .$$

These are all prime numbers and we have found the prime factorisation of 368.
In a more compact form we write, $368 = 2^4 \times 23$.

(ii). First note that 8085 is divisible by 5 as the last digit is a 5.

$$8085 \ = 5 \ \times \ 1617 \ (\ ^{*}\text{see below})$$
$$= 5 \times \ 3 \times \ 539$$
$$= 5 \times \ 3 \times \ 7 \times \ 77$$
$$= 5 \times \ 3 \times \ 7 \times \ 7 \times \ 11$$

These are all prime numbers and we have found the prime factorisation of 8085.

In a more compact form we write, $8085 \ = \ 3 \times \ 5 \times \ 7^2 \ \times \ 11$.

* We have used the following result:

A number is divisible 3 if the sum of its digits is divisible by 3.

The digit sum for 1617 is, $1 \ + \ 6 \ + \ 1 \ + \ 7 \ = \ 15$.

Since 15 is divisible by 3 the number 1617 is divisible by 3.

We have $1617 \ = \ 3 \times \ 539$.

(iii). First note that 2381400 is divisible by 100 as the last two digits are both 0.

$$2381400 \qquad = 100 \ \times \ 23814 \ (\ ^{**}\text{see below})$$
$$= \ 100 \ \times \ 9 \ \times \ 2646$$
$$= \ 100 \ \times \ 9 \ \times \ 9 \ \times \ 294$$
$$= \ 100 \ \times \ 9 \ \times \ 9 \ \times \ 3 \ \times \ 98$$
$$= \ 100 \ \times \ 9 \ \times \ 9 \ \times \ 3 \ \times \ 2 \ \times \ 49$$
$$= \ 100 \ \times \ 9 \ \times \ 9 \ \times \ 3 \ \times \ 2 \ \times \ 7 \ \times \ 7$$
$$= \ 2 \ \times \ 2 \ \times \ 5 \ \times \ 5 \ \times \ 9 \ \times \ 9 \ \times \ 3 \ \times \ 2 \ \times \ 7 \ \times \ 7$$
$$= \ 2 \ \times \ 2 \ \times \ 5 \ \times \ 5 \ \times \ 3 \ \times \ 3 \ \times \ 3 \ \times \ 3 \ \times \ 3 \ \times \ 2 \ \times \ 7 \ \times \ 7$$

These are all prime numbers and we have found the prime factorisation of 2381400.

In a more compact form we write, $2381400 \ = \ 2^3 \ \times \ 3^5 \ \times \ 5^2 \ \times \ 7^2$.

** We have used the following result:

A number is divisible 9 if the sum of its digits is divisible by 9.

The digit sum for 23814 is, $2 \ + \ 3 \ + \ 8 \ + \ 1 \ + \ 4 \ = \ 18$.

Since 18 is divisible by 9 the number 23814 is divisible by 9.

We have $23814 \ = \ 9 \ \times \ 2646$.

2.3.3. Least Common Multiple (LCM)

The **least common multiple** (LCM) of two non-zero integers a and b, is the **smallest** positive integer that is divisible by both a and b.

Example 7

Find the LCM of 9 and 12.

Solution

There are several different approaches to calculating the LCM.

(i). For "small" numbers like 9 and 12 the easiest method is to proceed as follows:
Write down several multiples of the smaller number:
Multiples of 9 are: 9, 18, 27, 36, 45, 54, . . .

Write down the multiples of the larger number until one of them is also a multiple of the smaller number:
Multiples of 12 are: 12, 24, 36, . . .

Now, 36 is also a multiple of 9 and so $\mathrm{lcm}(9, 12) = 36$.

(ii). An alternative approach uses prime factorisation:
Write the prime factorisation of each number in exponential form:
$$9 = 3 \times 3 = 3^2$$
$$12 = 2 \times 2 \times 3 = 2^2 \times 3.$$
Multiply all the factors with the highest powers to obtain the LCM.
$$\mathrm{lcm}(9, 12) = 3^2 \times 2^2 = 36.$$

2.3.4. Greatest Common Divisor (GCD)

The **greatest common divisor** (GCD) of two integers *a* and *b*, which are not both zero, is the **largest** integer that divides exactly into both numbers. The GCD is also known as the **highest common factor** (HCF).

Example 8

Find the GCD of 24 and 32.

Solution

There are several different approaches to calculating the GCD of two numbers.

(i). For "small" numbers like 24 and 32 the easiest method is to proceed as follows:

Write down the factors of the smaller number, starting from the largest factor.
The factors of 24 are: 24, 12, 8, 6, 3, 2, 1.

Write down the factors of the larger number, starting from the largest factor.
The factors of 32 are: 32, 16, 8, 4, 2, 1.

Reading from left to right, the first factor of the smaller number that is also a factor of the larger number is 8 and so 8 is defined to be the GCD of 24 and 32.

Hence, $\gcd(24, 32) = 8$.

(ii). An alternative approach uses prime factorisation.

Determine the prime factorisation of each number:

$$24 = 2 \times 2 \times 2 \times 3$$
$$32 = 2 \times 2 \times 2 \times 2 \times 2$$

List the factors common to both and multiply them together. The number 2 appears three times in each prime factorisation and so

$$\gcd(24, 32) = 2 \times 2 \times 2 = 8.$$

The methods in the previous example work well when the integers are relatively small but are inefficient as the numbers become larger. A more systematic approach for calculating the GCD of two positive integers is provided by the **Euclidean algorithm**, which is named after the ancient Greek mathematician Euclid. We do not consider the Euclidean algorithm in this unit.

2.3.5. Relatively Prime Integers

Two integers a and b are said to be **relatively prime** (or **coprime** or **mutually prime**) if they have no common factors other than 1, i.e. $\gcd(a, b) = 1$.

Example 9

(i). The numbers 175 and 256 are relatively prime as $\gcd(256, 175) = 1$.

(ii). Two prime numbers, a and b, are always relatively prime as $\gcd(a, b) = 1$.

Before moving on to look at modular arithmetic we present a useful theorem relating the GCD and LCM.

Theorem

Let a and b be two positive integers (natural numbers) then

$$\gcd(a, b) \times \text{lcm}(a, b) = a \times b.$$

In words, the product of the LCM and GCD of two (or more) natural numbers is equal to the product of the numbers.

Example 10

(i). $\gcd(24, 32) = 8$ and $\text{lcm}(24, 32) = 96$.
$\gcd(24, 32) \times \text{lcm}(24, 32) = 8 \times 96 = 768$.
Also, $24 \times 32 = 768$.

(ii). $\text{lcm}(16, 28) = \dfrac{16 \times 28}{\gcd(16, 28)} = \dfrac{448}{4} = 112$.

3. Modular Arithmetic

Modular arithmetic, or **clock arithmetic** as it is sometimes known, is a special type of arithmetic involving integers and features in branches of mathematics such as number theory and abstract algebra. Modular arithmetic plays an important role in cryptography where for example, it is used to reduce calculations involving large and very large integers to calculations that involve smaller integers.

In modulo m arithmetic all integers are replaced by their remainders after division by m. For example, if 8 is divided by 6 the remainder is 2. Here 6 is called the **modulus** and we write 8 (mod 6) = 2. We can perform this calculation for any number:

$$0 \ (\text{mod } 6) = 0$$
$$1 \ (\text{mod } 6) = 1$$
$$2 \ (\text{mod } 6) = 2$$
$$3 \ (\text{mod } 6) = 3$$
$$4 \ (\text{mod } 6) = 4$$
$$5 \ (\text{mod } 6) = 5$$
$$6 \ (\text{mod } 6) = 0$$
$$7 \ (\text{mod } 6) = 1$$
$$8 \ (\text{mod } 6) = 2, \text{ etc.}$$

Every time we reach a multiple of 6 we start counting from 0 again.

The set of integers modulo 6 is denoted: $\mathbb{Z}_6 = \{\, 0,\ 1,\ 2,\ 3,\ 4,\ 5 \,\}$.

In general, the set of integers modulo m is defined as: $\mathbb{Z}_m = \{\, 0,\ 1,\ 2,\ 3,\ .\ \ .\ \ .,\ m-1 \,\}$.

We actually encounter modular arithmetic every day when we tell the time as clocks work modulo 12 or 24 for hours and modulo 60 for minutes and seconds. Hence the term clock arithmetic.

Modular arithmetic allows standard mathematical calculations such as addition, subtraction and multiplication. Division by certain numbers is also possible but is not considered here. When the modulus is relatively small we can construct tables for arithmetic operations. These tables are also known as Cayley tables after the British mathematician Arthur Cayley (1821 – 1895). In the following sections we look at creating Cayley addition and multiplication tables.

3.1. Addition Tables

To create the addition table for $\mathbb{Z}_m$:

- Write the numbers from 0 to $m - 1$ in the borders of the table, i.e. in the top row and leftmost column.
- Add these row and column numbers together to obtain their sum.
- Divide the sum by the modulus (m) to obtain the remainder which is the answer.
- Enter the answer at the intersection of the appropriate row and column.

Example 11

The process is illustrated to generate the bottom row (highlighted) of the addition table for $\mathbb{Z}_6$. To obtain the entries in the bottom row, add 5 to each value that appears as a column header. Write the result modulo 6 in the appropriate cell where the row and column intersect.

- $(5 + 0)(\bmod 6) \equiv 5(\bmod 6) = 5$
- $(5 + 1)(\bmod 6) \equiv 6(\bmod 6) = 0$
- $(5 + 2)(\bmod 6) \equiv 7(\bmod 6) = 1$
- $(5 + 3)(\bmod 6) \equiv 8(\bmod 6) = 2$
- $(5 + 4)(\bmod 6) \equiv 9(\bmod 6) = 3$
- $(5 + 5)(\bmod 6) \equiv 10(\bmod 6) = 4$.

The remaining entries are generated in a similar manner to give the table:

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Addition Table for $\mathbb{Z}_6$

3.1.1. Additive Inverses

- The **additive inverse** of an integer a is the integer b such that $(a + b)(\bmod m) = 0$.
- Every integer in a modular arithmetic system has an additive inverse.
- In the addition table for $\mathbb{Z}_6$ we see that $(4 + 2)(\bmod 6) = 0$ and so the additive inverse of 4 in $\mathbb{Z}_6$ is 2. Conversely, the additive inverse of 2 in $\mathbb{Z}_6$ is 4.

3.2. Multiplication Tables

To create the multiplication table for $\mathbb{Z}_m$:

- Write the numbers from 0 to $m - 1$ in the borders of the table, i.e. in the top row and leftmost column.
- Multiply each pair of row/column numbers together to obtain their product.
- Divide the product by the modulus (m) to obtain the remainder which is the answer.
- Enter the answer at the intersection of the appropriate row and column.

Example 12

The process is illustrated to generate the bottom row (highlighted) of the multiplication table for $\mathbb{Z}_6$. To obtain the entries in the bottom row, multiply each value that appears as a column header by 5 and write the result modulo 6 in the appropriate cell where row and column intersect.

- $(5 \times 0) (\bmod 6) \equiv 0 (\bmod 6) = 0$
- $(5 \times 1) (\bmod 6) \equiv 5 (\bmod 6) = 5$
- $(5 \times 2) (\bmod 6) \equiv 10 (\bmod 6) = 4$
- $(5 \times 3) (\bmod 6) \equiv 15 (\bmod 6) = 3$
- $(5 \times 4) (\bmod 6) \equiv 20 (\bmod 6) = 2$
- $(5 \times 5) (\bmod 6) \equiv 25 (\bmod 6) = 1$

The remaining entries are generated in a similar manner to give the table:

$\times_6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Note that alternative notation for the multiplication of two integers modulo m is $a \times_m b$ and represents the remainder on division of $a \times b$ by m. For example, $5 \times_6 4 = 2$.

3.2.1. Multiplicative Inverses

- Let a, b and m be positive integers. The **multiplicative inverse** of a number a is defined to be the number b such that $ab \, (\bmod \, m) = 1$.

- A multiplicative inverse will exist only if a and b are relatively prime.

- The number 0 does not have a multiplicative inverse. Division by zero is not defined.

- We usually denote the inverse of an integer a as a^{-1} although some texts use, $\overline{a}$.

- A number with a multiplicative inverse will contain a '1' in its row at the column corresponding to the multiplicative inverse.

Example 13

Referring to the multiplication table above for $\mathbb{Z}_6$.

(i). The only two numbers with a multiplicative inverse modulo 6 are 1 and 5, i.e. the only two numbers in $\mathbb{Z}_6$ that are relatively prime with 6.

(ii). Note that 2 does not have a multiplicative inverse modulo 6 as there is no number which when multiplied by 2 yields 1. In other words there is no solution (x) to the equation $2x \, (\bmod \, 6) = 1$. Also, 3 and 4 have no inverses modulo 6

Note: In order for each non-zero element of $\mathbb{Z}_m$ to have a multiplicative inverse we must have that m is a prime number. The previous example showed that as 6 is not prime not every element in $\mathbb{Z}_6$ has an inverse.

Example 14

Construct a multiplication table for $\mathbb{Z}_5$ and identify which elements have an inverse.

$\times_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

- As 5 is a prime number every non-zero element in $\mathbb{Z}_5$ has a multiplicative inverse.
- The inverses are as follows: $1^{-1} = 1$, $2^{-1} = 3$, $3^{-1} = 2$ and $4^{-1} = 4$.

3.3. Subtraction Tables
We first of all look at how subtraction and in particular negative numbers are dealt with.

There are two possible outcomes of a subtraction:

A. The answer is positive
This case is treated exactly the same as before in that we divide the answer by the modulus and retain the remainder.

Example 15
(i). $(8 - 2) \pmod 9 = 6 \pmod 9 = 6.$

(ii). $(12 - 5) \pmod 3 = 7 \pmod 3 = 1.$

(iii). $(42 - 19) \pmod 8 = 23 \pmod 8 = 7.$

B. The answer is negative
Here we must add multiples of the modulus, *m*, to obtain a positive integer between 0 and *m*.

Example 16
(i). $(2 - 5) \pmod 7 = -3 \pmod 7 = 4$ since $-3 + 7 = 4.$

(ii). $(4 - 31) \pmod 5 = -27 \pmod 5 = 3$ since $-27 + (6 \times 5) = 3.$

(iii). $(11 - 29) \pmod 8 = -18 \pmod 8 = 6$ since $-18 + (3 \times 8) = 6.$

We can create a subtraction table, taking care to define the order of the subtraction, e.g. Row – Column. The subtraction table (mod 6) for $\mathbb{Z}_6$ is shown below.

$-_6$	0	1	2	3	4	5
0	0	5	4	3	2	1
1	1	0	5	4	3	2
2	2	1	0	5	4	3
3	3	2	1	0	5	4
4	4	3	2	1	0	5
5	5	4	3	2	1	0

4. Modulo 2 Matrix Arithmetic

We can now extend the concept of matrix multiplication, encountered earlier in the course, to operations on matrices in which the elements are all 0 or 1 with addition and multiplication carried out modulo 2.

Example 17

(i).
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1+0 & 1+1 \\ 1+1 & 1+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(ii).
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1\times0+0\times1 & 1\times1+0\times0 \\ 0\times0+1\times1 & 0\times1+1\times0 \end{bmatrix} = \begin{bmatrix} 0+0 & 1+0 \\ 0+1 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(iii).
$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0\times1+1\times0+1\times1 & 0\times1+1\times1+1\times0 & 0\times1+1\times1+1\times1 \\ 1\times1+0\times0+1\times1 & 1\times1+0\times1+1\times0 & 1\times1+0\times1+1\times1 \\ 1\times1+1\times0+0\times1 & 1\times1+1\times1+0\times0 & 1\times1+1\times1+0\times1 \end{bmatrix}.$$

$$= \begin{bmatrix} 0+0+1 & 0+1+0 & 0+1+1 \\ 1+0+1 & 1+0+0 & 1+0+1 \\ 1+0+0 & 1+1+0 & 1+1+0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

(iv).
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1\times1+1\times1 & 1\times1+1\times1 \\ 1\times1+1\times1 & 1\times1+1\times1 \end{bmatrix} = \begin{bmatrix} 1+1 & 1+1 \\ 1+1 & 1+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(v).
$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\times1+1\times1+0\times0 & 1\times0+1\times1+0\times1 \\ 0\times1+1\times1+0\times0 & 0\times0+1\times1+0\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+0 & 0+1+0 \\ 0+1+0 & 0+1+0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$
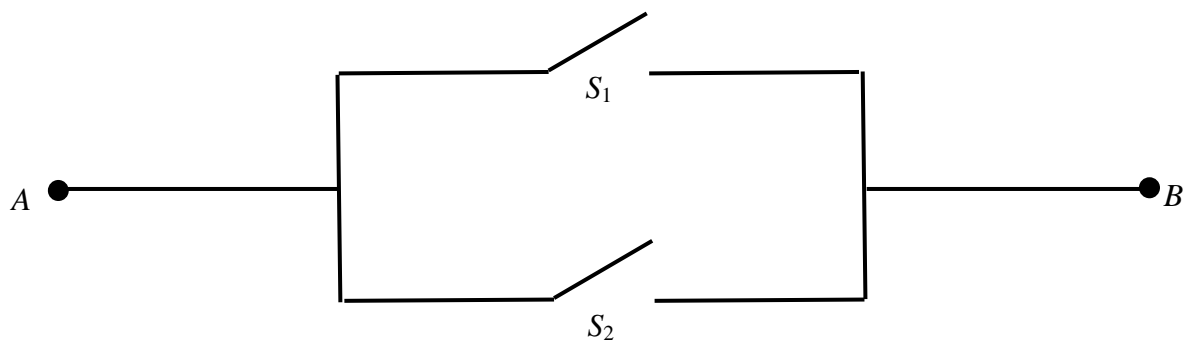
5. Boolean Algebra

Boolean Algebra was introduced by the English mathematician George Boole in 1854 and has many practical applications in the physical sciences including electrical engineering and computing. Essentially it is algebra suited to two-valued computer logic and enables algebraic manipulation of logical statements that occur in, for example, digital circuit theory.

A two element Boolean algebra is a set {0, 1} together with the binary operations sum and product, and the unary operation, complementation (also called negation). The two states, 1 and 0 are sometimes referred to as TRUE (T) and FALSE (F); 'ON' and 'OFF'; 'YES' and 'NO'; 'HIGH' and 'LOW', etc. The logical operators: sum, product and complementation are respectively associated with the OR, AND, and NOT operators in propositional logic.

In Boolean algebra $1 + 1 = 1$, just as 'TRUE OR TRUE' results in TRUE in propositional logic. Boolean addition corresponds to the logical OR function.

Consider the electric circuit shown in the schematic diagram below.

For current to flow from A to B we can have either one switch CLOSED or both switches CLOSED, i.e. we need S_1 OR S_2 CLOSED.

In Boolean algebra $1 \times 1 = 1$, just as 'TRUE AND TRUE' results in TRUE in propositional logic. Boolean multiplication corresponds to the logical AND function.

Note that Boolean multiplication is identical to standard arithmetic multiplication in that anything multiplied by 0 yields 0, and anything multiplied by 1 remains unchanged.

Consider the electric circuit shown in the schematic diagram below:
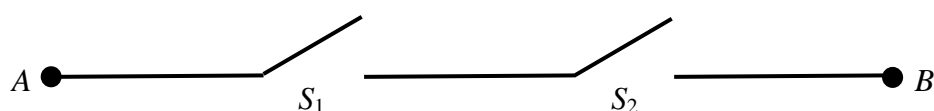
For current to flow from A to B we need to have both switches CLOSED, i.e. we need S_1 AND S_2 CLOSED.

6. Number Systems

This section introduces the binary, decimal and hexadecimal number systems that correspond to the bases, 2, 10 and 16 respectively. A point to note here is that the number of digits used by a number system is equal to the base of the system.

6.1. Base 10 (Decimal)

We are all familiar with the decimal, or base-10, system for counting as we have been using it most of our lives. In the base-10 system all numbers are represented by combinations of the ten **digits** between 0 and 9. If we start counting from 0, in due course, we will reach the number 9 and, as we have no unused single digits left, we increment the "tens" digit by one to give the number "ten", i.e. 10. Continuing to count we eventually reach 99 and, as we have no unused double digits left, we add one to the "hundreds" digit, giving 100.

Suppose for example, we continue to count until we reach the number 3472, say. We can tabulate this information as follows:

Place-Holder	10^3	10^2	10^1	10^0
Weight	1000	100	10	1
Decimal digit	3	4	7	2

In the decimal system the place-holder for each digit is a power of 10 so that moving from right to left, in the table, corresponds to an increase in magnitude by a factor of 10 at every step. To obtain the numerical value we multiply each decimal digit by its column's weight and sum the values, i.e.

$$2 \times 10^0 + 7 \times 10^1 + 4 \times 10^2 + 3 \times 10^3$$
$$= 2 \times 1 + 7 \times 10 + 4 \times 100 + 3 \times 1000 = 2 + 70 + 400 + 3000 = 3472$$

This number is therefore made up from the sum of: 2 "ones", 7 "tens", 4 "hundreds" and 3 "thousands".

Sometimes decimal numbers are written with the subscript "10" to indicate a base 10 number, e.g. 3472_{10}.

6.2. Base 2 (Binary)

The binary, or base 2, number system uses the two digits 0 and 1 to represent numbers and is of particular importance in computing. In a computer's memory elements can be in one of two states, OFF or ON corresponding to the digits 0 and 1 respectively. These elements represent one **bi**nary digi**t** or **bit**. All internal processing and calculations in computing are done in binary. We have seen that in base 10 every number can be written as a weighted sum of powers of 10. In an analogous manner for base 2 we use a weighted sum of powers of 2 to express numbers. The place-holder for each digit is therefore a power of 2 and moving from right to left corresponds to an increase in magnitude by a factor of 2 at every step. For example, consider the following table.

Place-Holder	2^3	2^2	2^1	2^0
Weight	8	4	2	1
Binary digit	1	1	0	1

The binary number in the table, 1101 is sometimes written with the subscript "2" to indicate that it is a base 2 number, i.e. 1101_2.

To obtain the decimal representation of 1101_2 we multiply each binary digit by its column's weight and sum the values. Starting from the right,

$$1 \times 2^0 + 0 \times 2^1 + 1 \times 2^2 + 1 \times 2^3 = 1 + 0 + 4 + 8 = 13.$$

The number is therefore made up from the sum of 1 "ones", 1 "fours" and 1 "eights".

Hence, $1101_2 = 13_{10}$.

6.3. Base 16 (Hexadecimal)

The hexadecimal (often called hex) or base 16 number system uses sixteen symbols,

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F,

to represent numbers. The first ten digits are the same as in the decimal system while the remaining six, A to F, correspond to the numbers from 10 to 15 respectively.

A computer carries out all its operations in binary but as numbers become large the binary representation requires increasingly more digits (0's and 1's) and becomes difficult for humans to read and write. For this reason computers often display information, such as memory addresses, in hexadecimal as their format is more compact.

In base 16 we use a weighted sum of powers of 16 to express numbers. The place-holder for each digit is therefore a power of 16 and moving from right to left corresponds to an increase in magnitude by a factor of 16 at every step. Consider the table given below:

Place-Holder	16^3	16^2	16^1	16^0
Weight	4096	256	16	1
Hex digit	1	2	B	F

The hex number in the table, 12BF can be written with the subscript "16", to indicate that it is a base 16 number, i.e. $12BF_{16}$.

To obtain the decimal representation of $12BF_{16}$ we multiply each hex digit by its column's weight, noting that B represents 11 and F corresponds to 15, and sum the values, i.e.

$$15 \times 16^0 + 11 \times 16^1 + 2 \times 16^2 + 1 \times 16^3 = 15 + 176 + 512 + 4096 = 4799.$$

Hence, $12BF_{16} = 4799_{10}$.

6.4. Decimal to Binary to Hexadecimal Look-Up Table

The following table shows the decimal, 4-bit binary and hexadecimal representations of the numbers 0 to 15. You should familiarise yourself with these values.

Decimal (Base 10)	Binary (Base 2)	Hexadecimal (Base 16)
0	0000	0
1	0001	1
2	0010	2
3	0011	3
4	0100	4
5	0101	5
6	0110	6
7	0111	7
8	1000	8
9	1001	9
10	1010	A
11	1011	B
12	1100	C
13	1101	D
14	1110	E
15	1111	F

6.5. Converting Between Number Bases

In this section we look at converting integers between different number systems. While the main focus will be on the bases most commonly used in computing, i.e. 2 (binary), 10 (decimal) and 16 (hex) we also present some results for other bases including octal (base 8). The ability to convert back and forth between different bases is a fundamental skill required of anyone working in the area of computing.

A base number converter can be found here http://www.kaagaard.dk/service/convert.htm

6.5.1. Converting from Base 10 to Any Base

Converting from base 10 (decimal) to any other base is easy. Start with the decimal number to be converted and repeatedly divide by the new base number retaining the remainder at each step. We shall illustrate with some examples.

Example 18
(i). Base 10 (Decimal) to Base 2 (Binary)

Convert the decimal number 475 to a binary number.

Solution

Start by dividing 475 by 2 and keep the remainder. Repeat the process until we can no longer perform a division.

$$475 / 2 = 237, \qquad \text{remainder } 1$$
$$237 / 2 = 118, \qquad \text{remainder } 1$$
$$118 / 2 = 59, \qquad \text{remainder } 0$$
$$59 / 2 = 29, \qquad \text{remainder } 1$$
$$29 / 2 = 14, \qquad \text{remainder } 1$$
$$14 / 2 = 7, \qquad \text{remainder } 0$$
$$7 / 2 = 3, \qquad \text{remainder } 1$$
$$3 / 2 = 1, \qquad \text{remainder } 1$$
$$1 / 2 = 0, \qquad \text{remainder } 1$$

Now read the binary number from the bottom to the top: 111011011.

Hence, $475_{10} = 111011011_2$.

(ii). Base 10 (Decimal) to Base 16 (Hexadecimal)

Convert the decimal number 795 to a hex number.

Solution

Start by dividing 795 by 16 and keep the remainder. Repeat the process until we can no longer perform a division.

     795 / 16 = 49,   remainder 11 (= B in hex)

     49 / 16 = 3,   remainder 1

     3 / 16 = 0,   remainder 3

Now read the hex number from the bottom to the top: 31B.

Hence, $795_{10} = 31B_{16}$.

(iii). Base 10 (Decimal) to Base 8 (Octal)

Convert the decimal number 5361 to an octal number.

Solution

Start by dividing 5361 by 8 and keep the remainder. Repeat the process until we can no longer perform a division. The octal number system is similar to decimal except that it only uses the eight digits from 0 to 7.

     5361 / 8 = 670,   remainder 1

     670 / 8 = 83,   remainder 6

     83 / 8 = 10,   remainder 3

     10 / 8 = 1,   remainder 2

     1 / 8 = 0,   remainder 1

Now read the octal number from the bottom to the top: 12361. Hence, $5361_{10} = 12361_8$.

Note: Octal used to be popular when computers employed 12-bit, 24-bit or 36-bit words for data and addressing. However, as modern computers all use 16-bit, 32-bit or 64-bit words octal is rarely used nowadays.

(iv). Base 10 (Decimal) to Base 4 (Quaternary)

Convert the decimal number 679 to a base 4 number.

Start by dividing 679 by 4 and keep the remainder. Repeat the process until we can no longer perform a division.

$$679 / 4 = 169, \qquad \text{remainder } 3$$
$$169 / 4 = 42, \qquad \text{remainder } 1$$
$$42 / 4 = 10, \qquad \text{remainder } 2$$
$$10 / 4 = 2, \qquad \text{remainder } 2$$
$$2 / 4 = 0, \qquad \text{remainder } 2$$

Now read the base 4 number from the bottom to the top: 22213.

Hence, $679_{10} = 22213_4$.

(v). Base 10 (Decimal) to Base 5 (Quinary)

Convert the decimal number 5361 to a base 5 number.

Solution

Start by dividing 5361 by 5 and keep the remainder. Repeat the process until we can no longer perform a division.

$$5361 / 5 = 1072, \qquad \text{remainder } 1$$
$$1072 / 5 = 214, \qquad \text{remainder } 2$$
$$214 / 5 = 42, \qquad \text{remainder } 4$$
$$42 / 5 = 8, \qquad \text{remainder } 2$$
$$8 / 5 = 1, \qquad \text{remainder } 3$$
$$1 / 5 = 0, \qquad \text{remainder } 1$$

Now read the base 5 number from the bottom to the top: 132421.

Hence, $5361_{10} = 132421_5$.

6.5.2. Converting from Any Base to Base 10 (Decimal)

Converting to base 10 (decimal) from any other base is also fairly straightforward. We shall consider two methods.

Method 1 – Place Value Method

The first method is based on the "place values" of the digits in the number being converted.

Let $N_b = x_1 x_2 x_3 . \quad . \quad . x_m$ be a base b number with m digits.

To convert to base 10 we calculate as follows:

$$N_{10} = x_m b^0 + x_{m-1} b^1 + x_{m-2} b^2 + . \quad . \quad . + x_1 b^{m-1} \qquad \text{[note the ordering!]}$$

Method 2 – Shortcut Method

Alternatively, we can apply the following algorithm which is much faster to use as numbers become larger.

- Start with the leftmost digit and multiply it by the base, b.

- Add the next digit and multiply by the base.

- Repeat until the rightmost digit has been added (Do not multiply by the base).

- This value is the result.

Example 19
(i). Binary to Decimal
(a).   Convert the binary number 11001 to a decimal number.

Solution
Place-Value Method
The place values of digits in a binary number are powers of 2. To convert 11001 proceed as follows:

$$11001 = 1 \times 2^0 + 0 \times 2^1 + 0 \times 2^2 + 1 \times 2^3 + 1 \times 2^4$$
$$= 1 + 0 + 0 + 8 + 16 = 25.$$

Hence, $11001_2 = 25_{10}$.

As base 2 only uses the numbers 0 and 1 this approach essentially involves adding the non-zero place values together.

Shortcut Method

Alternatively, proceed as follows:

- Multiply the leftmost digit, 1, by the base, 2 giving 2.
- Write 2 in the middle row of the second column.
- Add, $1 + 2 = 3$ and write 3 at the foot of the second column.

$$
\begin{array}{c|ccccc}
 & 1 & 1 & 0 & 0 & 1 \\
 & & 2 & & & \\
\hline
 & & 3 & & & \\
\end{array}
$$

- Multiply 3 by the base, 2 to obtain 6.
- Write 6 in the middle row of the third column.
- Add, $0 + 6 = 6$ and write 6 at the foot of the third column.

$$
\begin{array}{c|ccccc}
 & 1 & 1 & 0 & 0 & 1 \\
 & & 2 & 6 & & \\
\hline
 & & 3 & 6 & & \\
\end{array}
$$

- Multiply 6 by the base, 2 to obtain 12.
- Write 12 in the middle row of the fourth column.
- Add, $0 + 12 = 12$ and write 12 at the foot of the fourth column.

$$
\begin{array}{c|ccccc}
 & 1 & 1 & 0 & 0 & 1 \\
 & & 2 & 6 & 12 & \\
\hline
 & & 3 & 6 & 12 & \\
\end{array}
$$

- Multiply 12 by the base, 2 to obtain 24.
- Write 24 in the middle row of the fifth column.
- Add the rightmost digit, $1 + 24 = 25$ and write 25 at the foot of the fifth column.

$$
\begin{array}{c|ccccc}
 & 1 & 1 & 0 & 0 & 1 \\
 & & 2 & 6 & 12 & 24 \\
\hline
 & & 3 & 6 & 12 & 25 \\
\end{array}
$$

- The rightmost digit has been added and so we stop here.

Hence, $11001_2 = 25_{10}$ as obtained earlier.

(b). Convert the binary number 11011101 to a decimal number.

Solution

Place-Value Method

From the right, adding the place values, corresponding to the non-zero digits, in 11011101 gives:

$$1 + 0 + 4 + 8 + 16 + 0 + 64 + 128 = 221.$$

Hence, $11001_2 = 221_{10}$.

Shortcut Method

Alternatively, we can write

1	1	0	1	1	1	0	1
	2	6	12	26	54	110	220
3	6	13	27	55	110	221

Hence, $11011101_2 = 221_{10}$

The shortcut method becomes more efficient the larger the binary number. It is also particularly useful as the base number increases in size and this is demonstrated in the following example.

(ii). Hexadecimal to Decimal

(a). Convert the hexadecimal number 3B2 to a decimal number.

Solution

Place-Value Method

The place values of digits in a hex number are powers of 16. To convert 3B2 to its decimal representation, starting from the right, multiply each digit in 3B2 by the appropriate power of 16.

$$3B2_{16} = 2 \times 16^0 + 11 \times 16^1 + 3 \times 16^2$$
$$= 2 \times 1 + 11 \times 16 + 3 \times 256$$
$$= 2 + 176 + 768 = 946.$$

Hence, $3B2_{16} = 946_{10}$.

Shortcut Method

Alternatively, we can use the shortcut method where we multiply by 16.

$$
\begin{array}{r|rrr}
 & 3 & \text{B} & 2 \\
 & & 48 & 944 \\
\hline
 & 59 & 946 \\
\end{array}
$$

Hence, $3B2_{16} = 946_{10}$.

(b). Convert the hexadecimal number 4BAE to a decimal number.

Solution
Place-Value Method

$$
\begin{aligned}
4BAE_{16} &= 14 \times 16^0 + 10 \times 16^1 + 11 \times 16^2 + 4 \times 16^3 \\
&= 14 \times 1 + 10 \times 16 + 11 \times 256 + 4 \times 4096 \\
&= 19374 .
\end{aligned}
$$

Hence, $4BAE_{16} = 19374_{10}$.

Shortcut Method

Alternatively, calculate as follows:

$$
\begin{array}{r|rrrr}
 & 4 & \text{B} & \text{A} & \text{E} \\
 & & 64 & 1200 & 19360 \\
\hline
 & 75 & 1210 & 19374 \\
\end{array}
$$

Hence, $4BAE_{16} = 19374_{10}$.

Example 20
We now apply the shortcut method to convert from bases, other than 2 and 16, to decimal.

(i). Octal (Base 8) to Decimal
Convert the octal number 7630 to a decimal number.

Solution
The place values of digits in an octal number are powers of 8 and so we need to multiply by 8 in our calculations.

$$
\begin{array}{r|rrrr}
& 7 & 6 & 3 & 0 \\
& & 56 & 496 & 3992 \\
\hline
& 62 & 499 & 3992
\end{array}
$$

Hence, $7630_8 = 3992_{10}$

(ii). Base 5 to Decimal
Convert the base 5 number 4102 to a decimal number.

Solution
The place values of digits in an base 5 number are powers of 5 and so we need to multiply by 5 in our calculations.

$$
\begin{array}{r|rrrr}
& 4 & 1 & 0 & 2 \\
& & 20 & 105 & 525 \\
\hline
& 21 & 105 & 527
\end{array}
$$

Hence, $4102_5 = 527_{10}$.

(iii). Base 7 to Decimal

Convert the base 7 number 21643 to a decimal number.

Solution

The place values of digits in a base 7 number are powers of 7 and so we need to multiply by 7 in our calculations.

$$
\begin{array}{c|ccccc}
2 & 1 & 6 & 4 & 3 \\
& 14 & 105 & 777 & 5467 \\
\hline
15 & 111 & 781 & 5470
\end{array}
$$

Hence, $21643_7 = 5470_{10}$

6.5.3. Converting from Hexadecimal to Binary

As hexadecimal is base $16 = 2^4$ and binary is base $2 = 2^1$ every digit in a hex number can be replaced by its four bit binary equivalent.

Example 21

Convert the hexadecimal number 3C7D to a binary number.

Solution

Replace each hexadecimal number with its 4-bit binary equivalent.

Hex	3	C	7	D
Binary	0011	1100	0111	1101

Hence, $3C7D_{16} = 0011\,1100\;0111\,1101_2$.

6.5.4. Converting from Binary to Hexadecimal
Example 22
Convert the binary number 1111 1100 0100 1110 to a hexadecimal number.

Solution
- Starting from the right-hand side-split the number into groups of four. If necessary pad on the left with zeros to obtain a group of four.

- Convert each group of four to its decimal equivalent using the binary placeholder weightings, i.e. 1, 2, 4 and 8. For example, in the table below, the group of four on the right gives, $8 + 4 + 2 + 0 = 14$. Hence, $1110_2 = 14_{10}$.

- Convert each decimal number to its hex equivalent, e.g. $14_{10} = E_{16}$.

Binary	1111	1100	0100	1110
Decimal	15	12	4	14
Hex	F	C	4	E

Hence, $1111\ 1100\ 0100\ 1110_2 = FC4E_{16}$.

Note: With a little practice you will be able to omit the 'convert to decimal' step and convert each group of four directly from binary to hex.

6.5.5. Conversions Between Other Bases
Here we present some examples of how the methods described earlier can be applied to conversions between other bases.

Example 23
(i). Octal to Binary
As octal is base $8 = 2^3$ and binary is base $2 = 2^1$ every digit in a octal number can be replaced by its three bit binary equivalent.

Convert the octal number 761 to a binary number.

Solution

Replace each octal digit with its 3-bit binary equivalent.

Octal	7	6	1
Binary	111	110	001

Hence, $761_8 = 111110001_2$.

(ii). Binary to Octal

Convert the binary number 1110101000101 to an octal number.

Solution

- Starting from the right hand side split the number into groups of three. If necessary pad on the left with zeros to obtain a group of three.

- Convert each group of three to its octal equivalent using the binary placeholder weightings, i.e. 1, 2 and 4. For example, on the right we have, $4 + 0 + 1 = 5$.
  Hence, $101_2 = 5_8$

Binary	001	110	101	000	101
Octal	1	6	5	0	5

Hence, $1110101000101_2 = 16505_8$.

(iii). Hexadecimal to Octal

Convert the hexadecimal number 8B6E to an octal number.

Solution

One method is to convert the hex number to binary and then convert from binary to octal.

- Write each hex digit as a four bit binary number.

Hex	8	B	6	E
Binary	1000	1011	0110	1110

- Starting from the right, split the binary representation into groups of three. Pad the leftmost triple with zeros if required.

Binary	001	000	101	101	101	110
Octal	1	0	5	5	5	6

Hence, $8B6E_{16} = 105556_8$.

(iv). Octal to Hexadecimal

Convert the octal number 6473 to a hex number.

Solution

All we have to do is reverse the process in the previous example.

- Write each octal digit as a three bit binary number.

Octal	6	4	7	3
Binary	110	100	111	011

- Starting from the right, split the binary representation into groups of four. Pad the leftmost group with zeros if required.

- Convert each binary number to its decimal equivalent, e.g. $1011_2 = 11_{10}$.
- Convert each decimal number to its hex equivalent, e.g. $11_{10} = B_{16}$.

Binary	1101	0011	1011
Decimal	13	3	11
Hex	D	3	B

Hence, $6473_8 = D3B_{16}$.

Factors, Multiples and Primes

Q1. Find all the factors of the following integers:

    (i).   30   **(ii).**   100   **(iii).**   73   **(iv).**   84.

Q2. Determine the first three multiples of the following integers:

    (i).   7   **(ii).**   10   **(iii).**   131   **(iv).**   84.

Q3. Find the prime factorisation of:

    (i).   84   **(ii).**   585   **(iii).**   1040   **(iv).**   188

    (v).   279   **(vi).**   23600   **(vii).**   4023.

Q4. Find all the prime factors of 49.

Q5. If 63 and 112 are two multiples of a particular number, and the number is not 1, what is the number?

Q6. Find the LCM of 28 and 42.

Q7. It can be shown that the product of the LCM and GCD of two or more numbers is equal to the product of the numbers. Use this fact to answer the following:

    (i).   Determine the LCM of 64 and 92 if their GCD is 4.

    (ii).   Determine the GCD of 78 and 102 if their LCM is 1326.

Modular Arithmetic

Q8. Simplify each of the following:

(i).	$(3 + 4) \bmod 5$		**(ii).**	$(8 + 9) \bmod 13$
(iii).	$(9 + 3) \bmod 12$		**(iv).**	$(7 + 6) \bmod 12$
(v).	$(5 \times 3) \bmod 7$		**(vi).**	$(2 \times 10) \bmod 11$
(vii).	$(5 \times 6) \bmod 7$		**(viii).**	$(12 \times 11) \bmod 17$

Q9. Simplify each of the following:

(i).	$(5 - 3) \bmod 7$	**(ii).**	$(3 - 5) \bmod 7$	**(iii).**	$(4 - 8) \bmod 12$
(iv).	$(1 - 6) \bmod 7$	**(v).**	$(3 - 4) \bmod 5$	**(vi).**	$(1 - 10) \bmod 12$

Q10. Construct the Cayley addition table for $\mathbb{Z}_5$.

Q11. **(i).** Construct the Cayley multiplication table for $\mathbb{Z}_4$.

(ii). In $\mathbb{Z}_4$ find 2^{-1} and 3^{-1} if they exist. If not, explain why not.

(iii). Evaluate the following in $\mathbb{Z}_4$:

    (a). $(2 \times 3) + 3$ **(b).** $2 - 3 - 3$ **(c).** $(1 - 3) \times 3$

Q12. **(i).** Construct the Cayley multiplication table for $\mathbb{Z}_7$.

(ii). In $\mathbb{Z}_7$, find 4^{-1} and 6^{-1} if they exist. If not, explain why not.

(iii). Evaluate the following in $\mathbb{Z}_7$.

    (a). $(1 - 6) \times 3$ **(b).** $(1 - 4) \times 4$.

Q13. State the property of m that ensures that each element of $\mathbb{Z}_m$ has a multiplicative inverse.

Q14. Construct the subtraction table for $\mathbb{Z}_5$.

Q15. Construct Cayley tables for addition and multiplication modulo 2.

Modulo 2 Matrix Arithmetic

Q16. Where possible evaluate sums and products, using modulo 2 arithmetic, of the following pairs of matrices.

(i). $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$
(ii). $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$,
(iii). $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

Number Systems

Q17. Convert the following decimal numbers to their:

(i). binary representations;   **(ii).** hexadecimal representations.

(a). 39	**(b).** 73	**(c).** 100
(d). 359	**(e).** 111	**(f).** 733
(g). 1234	**(h).** 2012	**(j).** 2017.

Q18. Convert the following binary numbers to their:

(i). decimal representations;   **(ii).** hexadecimal representations

(a). 11111111	**(b).** 101011	**(c).** 111000111
(d). 1010010	**(e).** 111011100011	**(f).** 10011010101
(g). 100010001000	**(h).** 110100111.	

Q19. Convert the following hexadecimal numbers to their:

(i). decimal representations;   **(ii).** binary representations.

(a). BD3	**(b).** EEE	**(c).** 32
(c). 50	**(d).** ABC	**(e).** BBC1
(f). 2FC	**(g).** AA.	

Q20. Convert the following octal numbers to their:

(i). decimal representations;   **(ii).** binary representations;
(iii). hexadecimal representations.

(a). 7777	**(b).** 1234	**(c).** 1000.

Answers/Solutions to Tutorial Questions

S1. **(i).** The factors of 30 are: 1, 2, 3, 5, 6, 10, 15, and 30.

(ii). The factors of 100 are: 1, 2, 4, 5, 10, 20, 25, 50, 100.

(iii). The factors of 73 are: 1 and 73 (73 is a prime number).

(iv). The factors of 84 are: 1, 2, 3 ,4, 6, 7, 12, 14, 21, 28, 41, 84.

S2. **(i).** The first three multiples of 7 are: 7, 14 and 21.

(ii). The first three multiples of 10 are 10, 100 and 1000.

(iii). The first three multiples of 131 are 131, 262 and 393.

(iv). The first three multiples of 999 are 999, 1998 and 2997.

S3. **(i).** $84 = 2 \times 2 \times 3 \times 7$ **(ii).** $585 = 3 \times 3 \times 5 \times 13$

(iii). $1040 = 2 \times 2 \times 2 \times 2 \times 5 \times 13$ **(iv).** $188 = 2 \times 2 \times 47$

(v). $279 = 3 \times 3 \times 31$

(vi).
$$23600 = 100 \times 236$$
$$= 10 \times 10 \times 236$$
$$= 2 \times 5 \times 2 \times 5 \times 236$$
$$= 2 \times 5 \times 2 \times 5 \times 2 \times 118$$
$$= 2 \times 5 \times 2 \times 5 \times 2 \times 2 \times 59$$

(vii).
$$4023 = 3 \times 1341$$
$$= 3 \times 3 \times 447$$
$$= 3 \times 3 \times 3 \times 149$$

S4. $49 = 7 \times 7$. The only prime factor of 49 is 7.

S5. Determine the prime factorisation of each number:
$$63 = 3 \times 3 \times 7$$
$$112 = 2 \times 2 \times 2 \times 2 \times 7$$
List the factors common to both and multiply them together.
Only 7 is common to both and so GCD(63, 112) = 7.

S6. Multiples of 28 are: 28, 56, 84, 112, . . .
Multiples of 42 are: 42, 84, 126, . . .
Hence, LCM(28, 42) = 84.

S7. **(i).** $\text{lcm} \left(64, 92 \right) = \dfrac{64 \times 92}{\gcd \left(64, 92 \right)} = \dfrac{5888}{4} = 1472$.

(ii). $\gcd \left(78, 102 \right) = \dfrac{78 \times 102}{\text{lcm} \left(78, 102 \right)} = \dfrac{7956}{1326} = 6$.

Modular Arithmetic

S8. **(i).** 2   **(ii).** 4   **(iii).** 0   **(iv).** 1
     (v). 1   **(vi).** 9   **(vii).** 2   **(viii).** 13

S9. **(i).** 2   **(ii).** 5   **(iii).** 8   **(iv).** 2
     (v). 4   **(vi).** 3

S10. Cayley Addition Table for $\mathbb{Z}_5$

$+_5$	**0**	**1**	**2**	**3**	**4**
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

S11. **(i)** Cayley multiplication table for $\mathbb{Z}_4$.

$\times_4$	**0**	**1**	**2**	**3**
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

(ii). Note that 2^{-1} does not exist in $\mathbb{Z}_4$, as in $\mathbb{Z}_4$ there is no x such that $2x \equiv 1 \,(\text{mod}\, 4)$. From the table $3^{-1} = 3$.

(iii). **(a)** $(2 \times 3) + 3 \,(\text{mod}\, 4) = 9 \,(\text{mod}\, 4) = 1$
(b). $(2 - 3 - 3)\,(\text{mod}\, 4) = -4\,(\text{mod}\, 4) = 0$
(c). $(1 - 3) \times 3\,(\text{mod}\, 4) = -6\,(\text{mod}\, 4) = 2$

S12. **(i).** Cayley multiplication table for $\mathbb{Z}_7$.

$\times_7$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

(ii). From the table $4^{-1} = 2$ and $6^{-1} = 6$.

(iii). **(a)** $(1 - 6) \times 3\,(\text{mod}\, 7) = -15\,(\text{mod}\, 7) = 6$

(b) $(1 - 4) \times 4\,(\text{mod}\, 7) = -12\,(\text{mod}\, 7) = 2$

S13. Each element of $\mathbb{Z}_m$ has a multiplicative inverse only if m is prime.

S14. Cayley subtraction table for $\mathbb{Z}_5$.

$-$	0	1	2	3	4
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

S15. Cayley addition table for $\mathbb{Z}_2$:

$+_2$	**0**	**1**
0	0	1
1	1	0

Cayley multiplication table for $\mathbb{Z}_2$:

$\times_2$	**0**	**1**
0	0	0
1	0	1

Modulo 2 Matrix Arithmetic

S12. **(i).**
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1+1 & 0+0 \\ 0+1 & 1+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1\times1+0\times1 & 1\times0+0\times1 \\ 0\times1+1\times1 & 0\times0+1\times1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(ii).
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+1 & 1+1 & 1+1 \\ 1+0 & 1+1 & 1+1 \\ 1+1 & 1+0 & 1+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\times1+1\times0+1\times1 & 1\times1+1\times1+1\times0 & 1\times1+1\times1+1\times1 \\ 1\times1+1\times0+1\times1 & 1\times1+1\times1+1\times1 & 1\times1+1\times1+1\times1 \\ 1\times1+1\times0+1\times1 & 1\times1+1\times1+1\times0 & 1\times1+1\times1+1\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+1 & 1+1+0 & 1+1+1 \\ 1+0+1 & 1+1+0 & 1+1+1 \\ 1+0+1 & 1+1+0 & 1+1+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(iii). Cannot add these matrices as they have different sizes.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1\times0+0\times1 & 1\times1+0\times1 \\ 0\times0+0\times1 & 0\times1+0\times1 \\ 1\times0+0\times1 & 1\times1+0\times1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Number Systems

S17. **(i). Decimal to Binary**

(a).	100111	**(b).**	1001001	**(c).**	1100100
(d).	101100111	**(e).**	1101111	**(f).**	1011011101
(g).	10011010010	**(h).**	11111011100	**(j).**	11111100001.

(ii). Decimal to Hexadecimal

(a).	27	**(b).**	49	**(c).**	64
(d).	167	**(e).**	6F	**(f).**	2DD
(g).	4D2	**(h).**	7DC	**(j).**	7E1.

S18. **(i). Binary to Decimal**

(a).	255	**(b).**	43	**(c).**	455
(d).	82	**(e).**	3811	**(f).**	1237
(g).	2184	**(h).**	423.		

(ii). Binary to Hexadecimal

(a).	FF	**(b).**	2B	**(c).**	1C7	**(d).**	52
(e).	EE3	**(f).**	4D5	**(g).**	888	**(h).**	1A7.

S19. **(i). Hexadecimal to Decimal**

(a).	3027	**(b).**	3822	**(c).**	50	**(d).**	80
(e).	2748	**(f).**	48065	**(g).**	764	**(h).**	170.

(ii). Hexadecimal to Binary

(a).	101111010011	**(b).**	111011101110	**(c).**	110010
(d).	1010000	**(e).**	101010111100		
(f).	1011101111000001	**(g).**	1011111100	**(h).**	10101010.

S20. **(i).** **Octal to Decimal**

(a).	4095	**(b).**	668	**(c).**	512.

(ii). **Octal to Binary**

(a).	111111111111	**(b).**	1010011100	**(c).**	1000000000 .

(iii). **Octal to Hexadecimal**

(a).	FFF	**(b).**	29C	**(c).**	200 .