Glasgow Caledonian University

School of Computing, Engineering & Built Environment

Mathematics Summer School

Level 2 Entry – Engineering

&

Level 3 Entry – Computing

Vectors

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Scalars and Vectors

1). What are Scalars and Vectors?

As you are all aware, we can use numbers to measure or quantify physical quantities. For example, a thermometer may tell us that the temperature at a particular point in a room at a particular time is 24°C. A voltmeter may indicate a voltage of 9V across the terminals of a battery. Many quantities, such as temperature and voltage, require just a single number to specify or measure them; others may require more than one number. Relative position is a physical quantity that we can measure. We might say that Jack is 6.5m from Jill. If we do, then we haven't told the whole story. Is Jack 6.5m to Jill's right, to her left, behind her, above her, or what? To specify relative position completely we must give not only a distance, but a direction as well. This would require at least two numbers.

Basically, physical quantities that are specified by single numbers are termed **scalars**, while those that require more than one number are called **vectors**.

Examples of Scalars

- mass
- energy
- temperature
- length
- area (sometimes)
- volume
- density
- voltage
- current
- entropy

Examples of Vectors

- relative position
- area (sometimes)
- displacement
- velocity
- acceleration
- force
- moment of force (torque)
- linear momentum
- angular momentum

2). Representing a Vector Mathematically – Polar Form

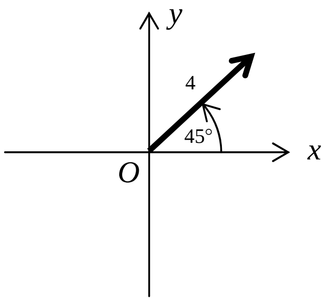
In most practical cases a vector will be a quantity specified by a **magnitude** and a **direction**. The simplest way to represent such a composite quantity is by an arrow whose length is scaled to correspond to the magnitude and whose orientation represents the vector's direction of action. In general the arrow will lie in 3-dimensional space, but to keep things simple (just for the moment) we shall work in 2 dimensions, like a map.

There are various ways to specify the direction of action, for example compass points or map bearings if we are dealing with relative position or displacement. We shall use the mathematical convention of drawing an arrow on an *Oxy* axes system and recording the angle it makes with the positive direction of the *x*-axis (with positive angles measured anticlockwise and negative angles, clockwise).

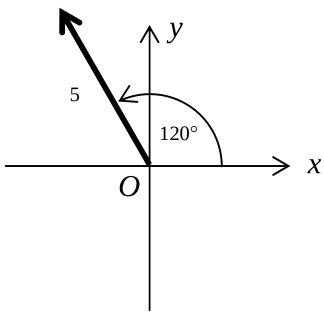
Notationally, we usually denote vectors by an underlined letter to distinguish them from scalars, e.g. $\underline{v}$, $\underline{v}_1$, $\underline{F}$, etc. Algebraically, we group the magnitude and direction as a bracketed ordered pair of the form $(r\ ,\ \theta°)$, the so-called **polar form** of a two-dimensional vector.

Note: We sometimes use $|\ \underline{v}\ |$ to denote the magnitude of a vector $\underline{v}$.

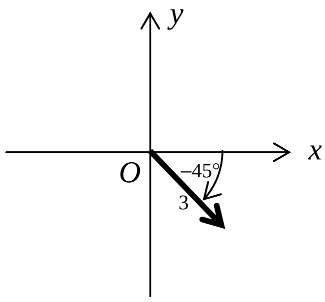
Examples



$\underline{v}\ =\ (4\ ,\ 45°)$

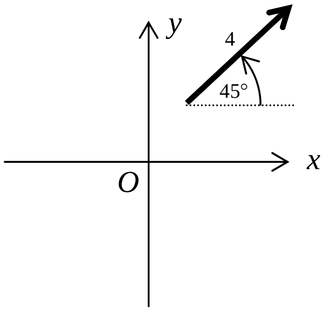


$\underline{v}\ =\ (5\ ,120°)$

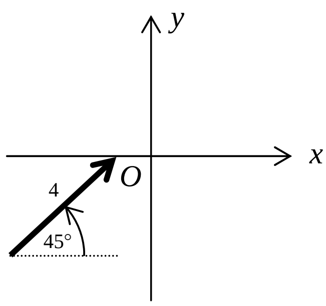


$\underline{v}\ =\ (3\ ,-45°)$

The arrows don't necessarily have to emanate from the origin:



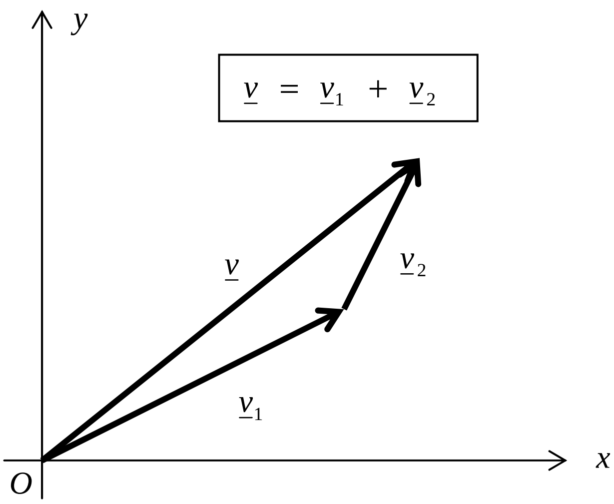
$\underline{v}\ =\ (4\ ,\ 45°)$



$\underline{v}\ =\ (4\ ,\ 45°)$

3). Vector Addition and Subtraction

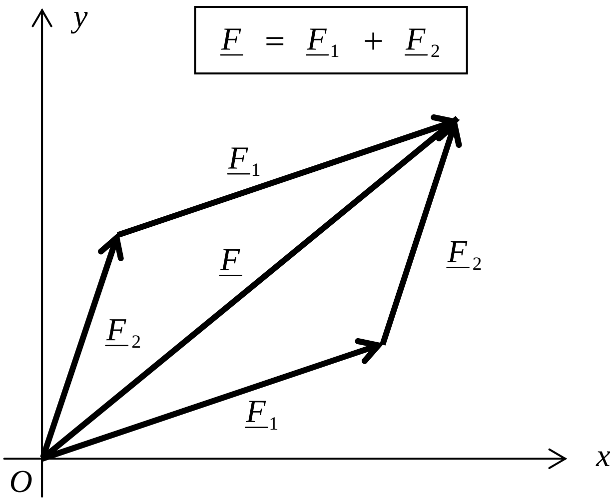
Consider two displacement vectors $\underline{v}_1 = (4\,,\,30°)$ and $\underline{v}_2 = (2\,,\,60°)$. We define the sum of these two vectors as the net resultant displacement of one following on from the other.



It is easily demonstrated that the order of addition does not matter. That is,

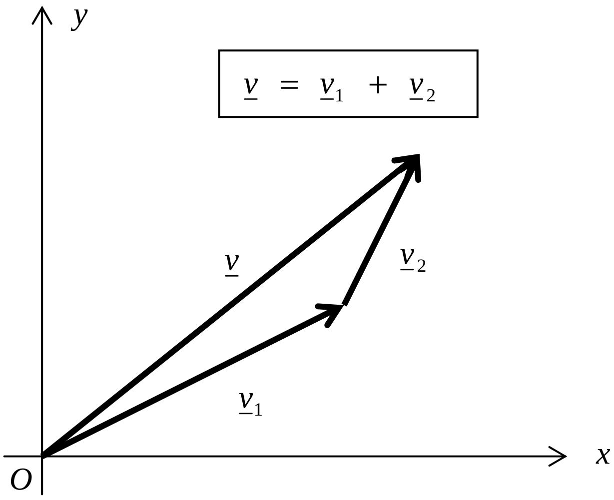
$$\underline{v} = \underline{v}_1 + \underline{v}_2 = \underline{v}_2 + \underline{v}_1 \ .$$

This definition, sometimes called the **triangle law** for the addition of two vectors, is fairly natural when thinking in terms of displacement, but the same definition proves to be applicable for other types of vector. Consider to forces, $\underline{F}_1$ and $\underline{F}_2$ acting at the origin. The net effect of the two forces is given by the vector sum, as defined above.



This is the **parallelogram rule** for adding two vectors, which is clearly equivalent to the triangle law.

Let’s look again at the first diagram illustrating vector addition:



Algebraically, we would like the vector notation to behave like scalar notation. That is,

$$\underline{v} = \underline{v}_1 + \underline{v}_2$$

should lead to

$$\underline{v}_1 = \underline{v} - \underline{v}_2 .$$

Writing this as

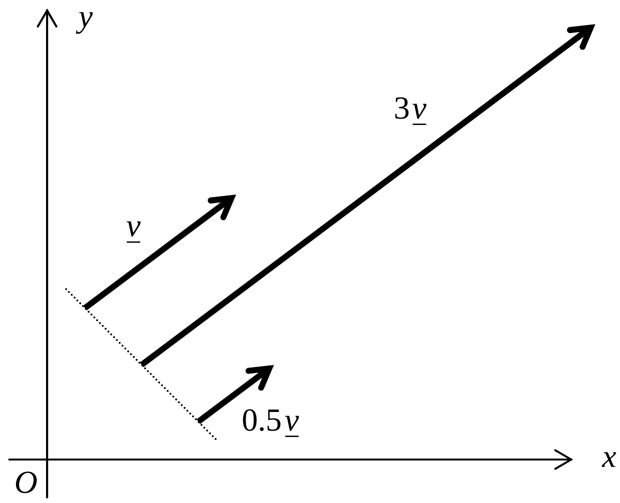
$$\underline{v}_1 = \underline{v} + (-\underline{v}_2)$$

and noting from the diagram that $\underline{v}_1$ can be interpreted as the net result of “moving” along $\underline{v}$, then <u>backwards</u> along $\underline{v}_2$, it should be obvious that $(-\underline{v}_2)$ is simply $\underline{v}_2$ with its direction reversed. This means that vector subtraction is just a special case of vector addition.

All this defines vector addition and subtraction geometrically, but does not give us an easy way of evaluating vector sums and differences. We can do such calculations using trigonometry and Pythagoras, but they aren’t straightforward calculations. Later we shall look at an alternative representation of two-dimensional vectors, one that makes the addition and subtraction of vectors dead easy. Before that, let us look at one other piece of vector arithmetic.

4). Multiplying a Vector by a Scalar

If we have a vector $\underline{v}$ and a positive scalar k, then we can define the product $k\underline{v}$ as a vector oriented in the same direction as $\underline{v}$ with a magnitude k times that of $\underline{v}$:

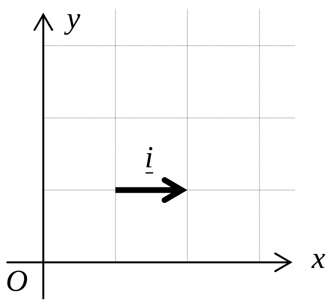


Note: Multiplying by a negative scalar would reverse the direction.

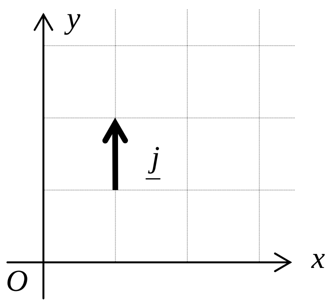
5). The Cartesian or Rectangular Component Form of a Vector

Using the definitions of vector addition and multiplication by a scalar, we can now develop an alternative algebraic representation of a two-dimensional vector that is easier to work with than the polar form.

A unit vector is a vector with magnitude equal to one unit. We introduce two unit vectors, one parallel to the x-axis and one parallel to the y-axis; these we denote by $\underline{i}$ and $\underline{j}$ respectively:

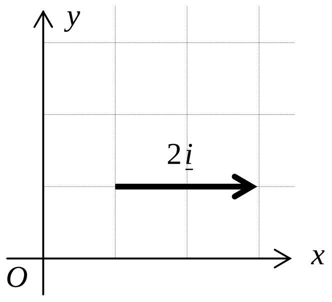


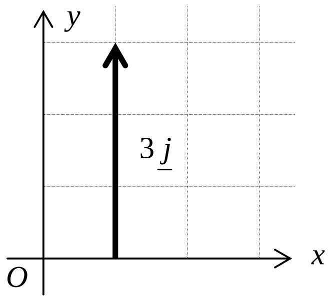
$$\underline{i} = (1, 0°)$$

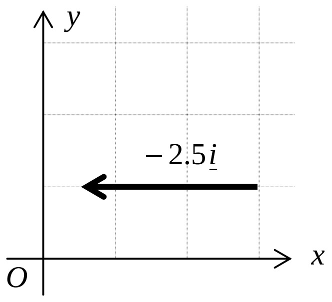


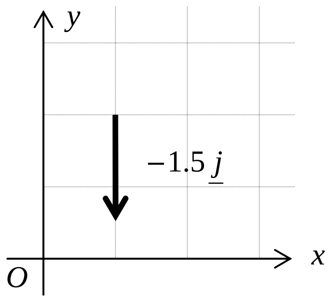
$$\underline{j} = (1, 90°)$$

This means that any vector parallel to one of the axes can be expressed as a scalar multiple of either $\underline{i}$ or $\underline{j}$:

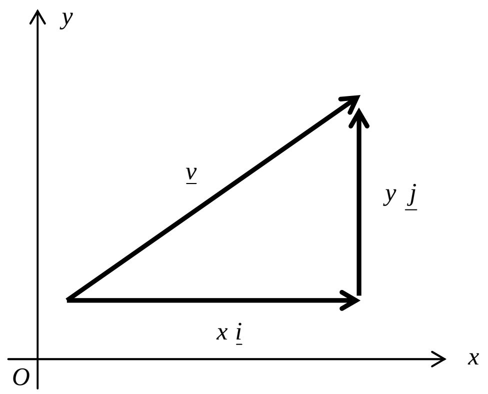








Bearing this and the definition of vector addition in mind, we can take any two-dimensional vector $\underline{v}$ and decompose it into the sum of two component vectors, one parallel to the *x*-axis and one parallel to the *y*-axis:



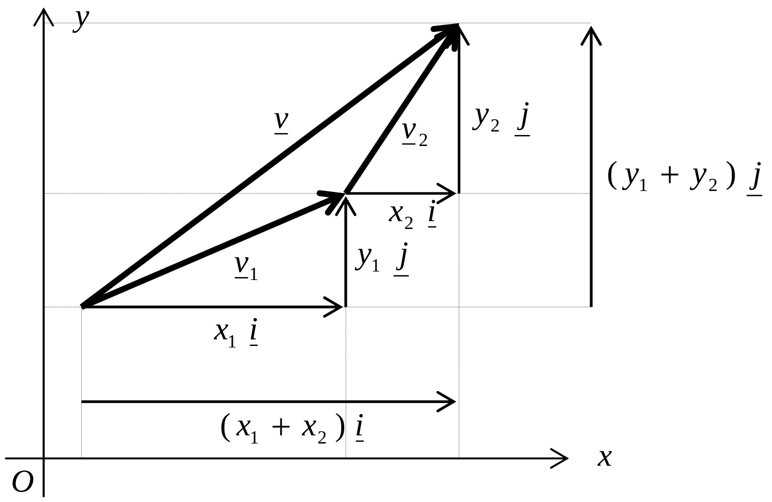
Algebraically we have

$$\underline{v} = x\,\underline{i} + y\,\underline{j} \quad .$$

The vector is completely specified by the two values x and y. These are the Cartesian or rectangular components of the vector and we can use these as an alternative to the polar components r and θ.

The advantage of the rectangular form over the polar is that vector arithmetic is easier. If we have two vectors, $\underline{v}_1 = x_1\,\underline{i} + y_1\,\underline{j}$ and $\underline{v}_2 = x_2\,\underline{i} + y_2\,\underline{j}$, then it is quite easy to see that

$$\underline{v} = \underline{v}_1 + \underline{v}_2 = (x_1 + x_2)\,\underline{i} + (y_1 + y_2)\,\underline{j}$$



Multiplying a vector by a scalar is also easy in rectangular form:

$$\underline{v} = x\,\underline{i} + y\,\underline{j}$$

$$k\,\underline{v} = (k\,x)\,\underline{i} + (k\,y)\,\underline{j}\ .$$

Convince yourself of this by sketching it out on paper; think of similar triangles!

Examples

(1). Given $\underline{v}_1 = 7\,\underline{i} + 4\,\underline{j}$ and $\underline{v}_2 = 2\,\underline{i} + 6\,\underline{j}$ determine

(i). $\underline{v}_1 + \underline{v}_2$ [Ans: $9\,\underline{i} + 10\,\underline{j}$]

(ii). $\underline{v}_1 - \underline{v}_2$ [Ans: $5\,\underline{i} - 2\,\underline{j}$]

(iii). $3\underline{v}_1$ [Ans: $21\,\underline{i} + 12\,\underline{j}$]

(iv). $3\underline{v}_1 + 2\underline{v}_2$ [Ans: $25\,\underline{i} + 24\,\underline{j}$]

Instead of always carrying the $\underline{i}$ and $\underline{j}$, we can use an abbreviated notation for the Cartesian form:

$$\underline{v} \;=\; (x\,,\,y)\,.$$

We have to be a little careful when using this notation since $(x\,,\,y)$ is also used for the coordinates of a point on the *Oxy* axes system. In most cases, the correct interpretation should be clear from the context. In this abbreviated form we have

$$\underline{v}_1 \;+\; \underline{v}_2 \;=\; (x_1 + x_2\;,\;y_1 + y_2)$$

$$\underline{v}_1 \;-\; \underline{v}_2 \;=\; (x_1 - x_2\;,\;y_1 - y_2)$$

$$k\,\underline{v} \;=\; (k\,x\,,\,k\,y)\;.$$

Using this notation, **Example (1)** would be set out as:

(1). Given $\underline{v}_1 \;=\; (7\,,\,4)$ and $\underline{v}_2 \;=\; (2\,,\,6)$ determine

(i). $\underline{v}_1 \;+\; \underline{v}_2$ [Ans: $(9\,,\,10)$]

(ii). $\underline{v}_1 \;-\; \underline{v}_2$ [Ans: $(5\,,\,-2)$]

(iii). $3\underline{v}_1$ [Ans: $(21\,,\,12)$]

(iv). $3\underline{v}_1 \;+\; 2\underline{v}_2$ [Ans: $(25\,,\,24)$]

6). The Relationship Between Polar and Cartesian (Rectangular) Forms

Polar Form: $\underline{v} \;=\; (r\,,\,\theta^\circ)$

Rectangular Form: $\underline{v} \;=\; (x\,,\,y)$

A combination of basic trigonometry and Pythagoras' Theorem gives the following conversion formulae:

Polar $\rightarrow$ Rectangular: $x \;=\; r\,\cos\theta^\circ$ $\qquad y \;=\; r\,\sin\theta^\circ$

Rectangular $\rightarrow$ Polar: $r \;=\; |\,\underline{v}\,| = \sqrt{x^2 + y^2}$ $\qquad \theta \;=\; \tan^{-1}\left(\dfrac{y}{x}\right)$.

The conversion "Polar $\rightarrow$ Rectangular" is quite straightforward, but care must be taken when applying "Rectangular $\rightarrow$ Polar", since the quadrant in which θ lies must be determined before evaluating the inverse tangent.

Examples

(2). (i). A vector has magnitude 2 and direction $210°$. Express the vector in its Cartesian component form.

$$\underline{v} = (r\,,\theta°) = (2\,,210°)$$

$$x = r\cos\theta° \qquad y = r\sin\theta°$$
$$= 2\cos(210°) \qquad = 2\sin(210°)$$
$$= -1.732 \qquad = -1$$

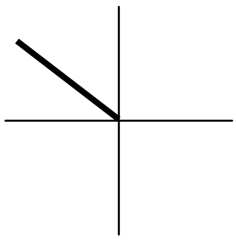
$$\underline{v} = -1.732\,\underline{i} - 1\,\underline{j} = (-1.732\,,-1)$$

(ii). Express the vector $\underline{v} = (-4\,,2)$ in polar form:

$$\underline{v} = (x\,,y) = (-4\,,2)$$

$$r = \sqrt{x^2 + y^2}$$
$$= \sqrt{(-4)^2 + 2^2}$$
$$= \sqrt{20}$$
$$= 4.472$$

Determine the quadrant for the angle ($x = -4$, $y = 2$) :



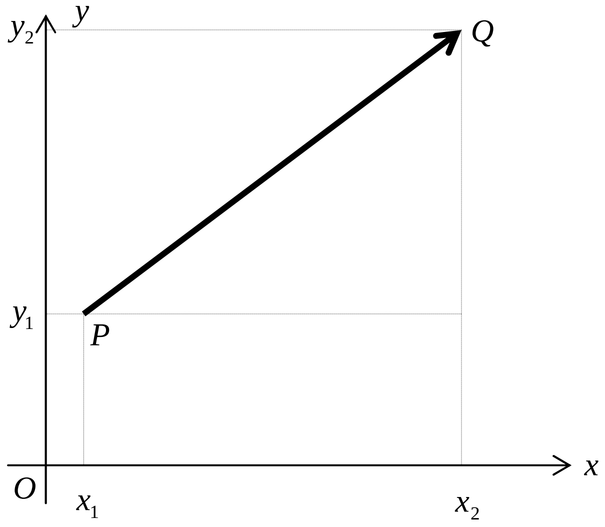
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$= \tan^{-1}\left(\frac{2}{-4}\right) \quad \text{[2nd quadrant angle]}$$
$$= 153.43°$$

$$\underline{v} = (4.472\,,153.43°)$$

Most calculators have these conversion formulae pre-programmed. Please refer to your own calculator's instruction booklet for information on how to implement these conversion processes or, if that fails, ask in the tutorial classes.

7). Relative Position Vectors in Rectangular Form

Suppose we have two **points** on an *Oxy*-axes system, $P(x_1\ ,\ y_1)$ and $Q(x_2\ ,\ y_2)$.

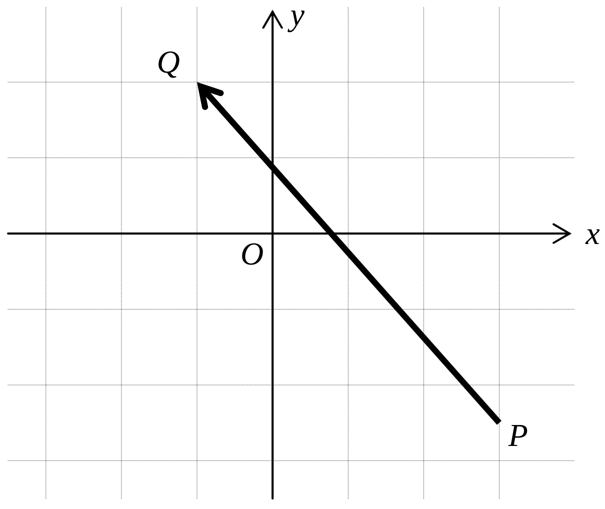


The position of Q relative to P is given by a two-dimensional vector $\overrightarrow{PQ}$ (note notation) whose Cartesian components are easily determined from the coordinates of the points:

$$\overrightarrow{PQ} = (x_2 - x_1\ ,\ y_2 - y_1)\ .$$

Example

(3). Determine the position of the point $Q(-1\ ,\ 2)$ relative to $P(3\ ,\ -2.5)$.



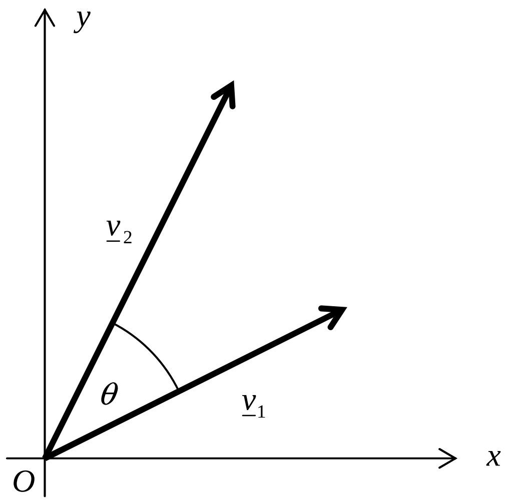
$$\overrightarrow{PQ} = (-1-3\ ,\ 2-(-2.5))$$

$$= \underline{\underline{(-4\ ,\ 4.5)}}$$

8). The Scalar Product of Two Vectors (Dot Product)

We now extend the arithmetic of vectors and introduce a way of multiplying two vectors together. There are, in fact, two ways of multiplying vectors together. In this section we shall look at the way that results in a scalar product, i.e. a number rather than another vector. Later we shall look at the other way which does result in another vector.

Consider two vectors $\underline{v}_1$ and $\underline{v}_2$ whose orientations differ by an angle θ:



We define their **scalar product** as

$$\underline{v}_1 \cdot \underline{v}_2 \;=\; |\underline{v}_1|\,|\underline{v}_2| \cos\theta \;.$$

Because we usually make a clear dot between the two vectors to distinguish this from the other form of multiplication that we shall see later, this is sometimes referred to as the **dot product**.

For two (two-dimensional) vectors expressed in polar form, this scalar product is given by

$$\underline{v}_1 \cdot \underline{v}_2 \;=\; r_1\, r_2 \,\cos(\theta_1 - \theta_2) \;.$$

Working in rectangular form it can be shown that

$$\underline{v}_1 \cdot \underline{v}_2 \;=\; x_1 x_2 \;+\; y_1 y_2 \;.$$

Examples

(4). (i). Determine the scalar (or dot) product of $\underline{v}_1 = (2\,,\,4)$ and $\underline{v}_2 = (3\,,\,5)$

$$\underline{v}_1 \cdot \underline{v}_2 \;=\; 2\times 3 \;+\; 4\times 5 \;=\; 26$$

(ii). Determine the scalar (or dot) product of $\underline{v}_1 = (3\,,\,-1)$ and $\underline{v}_2 = (-6\,,\,-2)$

$$\underline{v}_1 \cdot \underline{v}_2 \;=\; 3\times(-6) \;+\; (-1)\times(-2) \;=\; -16$$

(5). **(i).** Determine the angle of separation of the two vectors $\underline{v}_1 = (2\ ,\ 4)$ and $\underline{v}_2 = (3\ ,\ 5)$.

$$\underline{v}_1 \cdot \underline{v}_2 \;=\; 2 \times 3 \;+\; 4 \times 5 \;=\; 26$$

$$|\,\underline{v}_1\,| \;=\; \sqrt{2^2 + 4^2} \;=\; \sqrt{20}$$

$$|\,\underline{v}_2\,| \;=\; \sqrt{3^2 + 5^2} \;=\; \sqrt{34}\ .$$

From the definition of the dot product

$$\begin{aligned} \cos\theta \;&=\; \frac{\underline{v}_1 \cdot \underline{v}_2}{|\,\underline{v}_1\,|\,|\,\underline{v}_2\,|} \\ &=\; \frac{26}{\sqrt{20}\,\sqrt{34}} \\ &=\; 0.99705 \end{aligned}$$

and so the angle of separation is $\underline{\underline{\theta \;=\; 4.40^\circ}}$.

Note: Usually we have to take care with quadrants when we invert a trig function. However, the angle of separation of two vectors will always be between 0° and 180° (inclusive) and this is what a calculator cosine inversion will always give us.

(ii). Determine the angle of separation of the two vectors $\underline{v}_1 = (3\ ,\ -1)$ and $\underline{v}_2 = (-6\ ,\ -2)$.

$$\underline{v}_1 \cdot \underline{v}_2 \;=\; 3 \times (-6) \;+\; (-1) \times (-2) \;=\; -16$$

$$|\,\underline{v}_1\,| \;=\; \sqrt{3^2 + (-1)^2} \;=\; \sqrt{10}$$

$$|\,\underline{v}_2\,| \;=\; \sqrt{(-6)^2 + (-2)^2} \;=\; \sqrt{40}\ .$$

From the definition of the dot product

$$\begin{aligned} \cos\theta \;&=\; \frac{\underline{v}_1 \cdot \underline{v}_2}{|\,\underline{v}_1\,|\,|\,\underline{v}_2\,|} \\ &=\; \frac{-16}{\sqrt{10}\,\sqrt{40}} \\ &=\; -0.8 \end{aligned}$$

and so the angle of separation is $\underline{\underline{\theta \;=\; 143.13^\circ}}$.

The scalar product has a number of applications. Here are just a few:

(a). Test for orthogonality:

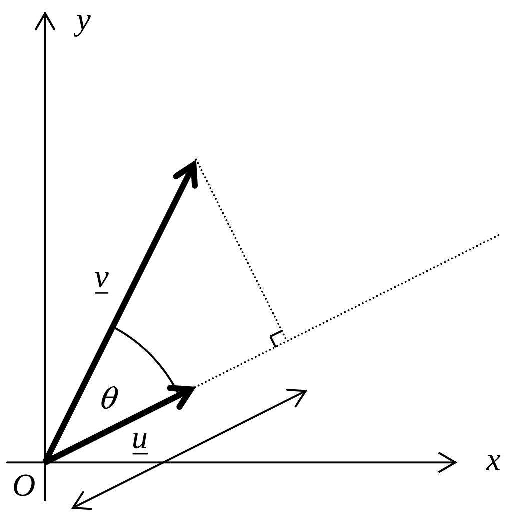
Two vectors are said to be orthogonal if they act at right-angles to each other. For $\underline{v}_1$ and $\underline{v}_2$ at right-angles we have

$$\underline{v}_1 \cdot \underline{v}_2 \;=\; |\,\underline{v}_1\,|\,|\,\underline{v}_2\,| \cos 90° \;=\; 0 \;.$$

This gives us a test for orthogonality; we look for a scalar product equal to zero.

(b). Component of a vector in a given direction:

We have already spoken about vectors having components parallel to the x and y axes. Using the dot product, we can determine the component of a vector in any given direction. Consider a vector $\underline{v}$ and a direction specified by a **unit vector** $\underline{u}$ (i.e. $|\,\underline{u}\,| = 1$).



The component of $\underline{v}$ in the direction of $\underline{u}$ is given by the length of the double-headed arrow in the above diagram. This is exactly the dot product

$$\underline{v} \cdot \underline{u} \;=\; |\,\underline{v}\,|\,|\,\underline{u}\,| \cos \theta \;=\; |\,\underline{v}\,| \cos \theta$$

since the magnitude of $\underline{u}$ equals 1 .

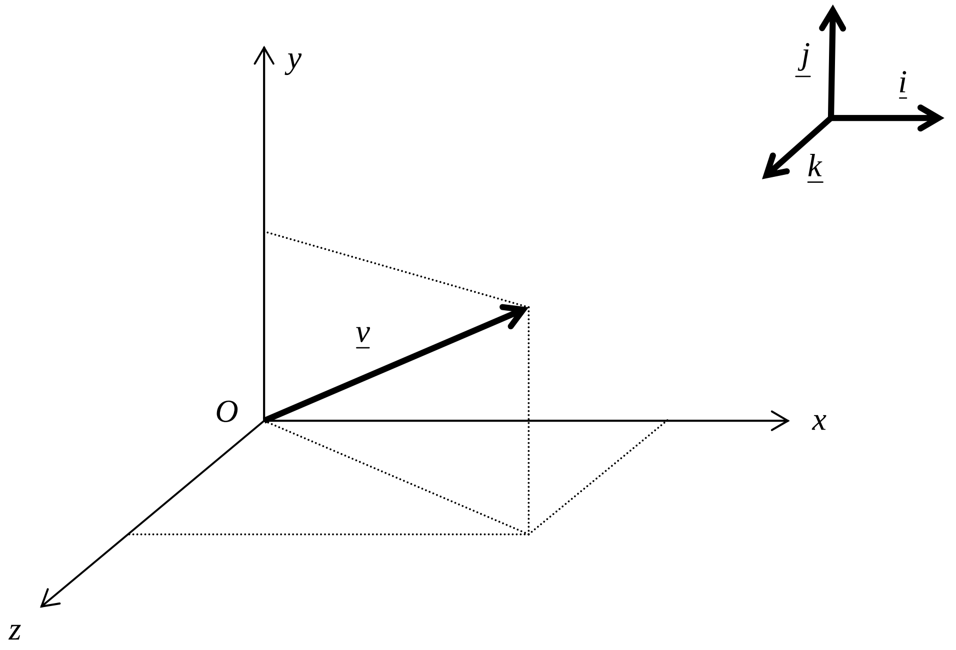
(c). Work done by a force:

Suppose an object is acted upon by several forces, $\underline{F}_1 , \underline{F}_2 , \ldots , \underline{F}_n$, and undergoes a displacement $\underline{d}$. The work done by each individual force is given by the dot product

$$\underline{F}_k \cdot \underline{d} \qquad , \qquad k \;=\; 1 , 2 , \ldots , n \;.$$

9). Extension to Three Dimensions

So far we have only considered two-dimensional vectors. The Cartesian form of a vector provides an easy and natural way of extending vectors into three dimensions. We take the *Oxy* axes system and include a third axis, a *z*-axis, perpendicular to both the *x* and *y* axes:



Denoting the unit vector parallel to the new *z*-axis by $\underline{k}$, any three-dimensional vector $\underline{v}$ can be decomposed into the sum of three component vectors, one in each of the axes directions:

$$\underline{v} \;=\; x\,\underline{i} \;+\; y\,\underline{j} \;+\; z\,\underline{k} \;=\; (x\,,\,y\,,\,z)\;\;.$$

Vector magnitudes, addition, subtraction, multiplication by scalars, and dot products follow naturally. For $\underline{v} = (x\,,\,y\,,\,z)$, $\underline{v}_1 = (x_1\,,\,y_1\,,\,z_1)$ and $\underline{v}_2 = (x_2\,,\,y_2\,,\,z_2)$ we have:

Magnitude:	$\lvert\,\underline{v}\,\rvert \;=\; \sqrt{x^2 + y^2 + z^2}$
Addition:	$\underline{v}_1 + \underline{v}_2 \;=\; (x_1 + x_2\,,\,y_1 + y_2\,,\,z_1 + z_2)$
Subtraction:	$\underline{v}_1 - \underline{v}_2 \;=\; (x_1 - x_2\,,\,y_1 - y_2\,,\,z_1 - z_2)$
Mult. by scalar:	$k\,\underline{v} \;=\; (k\,x\,,\,k\,y\,,\,k\,z)$
Scalar (dot) product:	$\underline{v}_1 \cdot \underline{v}_2 \;=\; x_1\,x_2 + y_1\,y_2 + z_1\,z_2\;\;.$

Note: The polar form of a vector does not extend as naturally into three dimensions as the rectangular form. There do exist alternative three-dimensional forms of vectors related to the polar form, however we do not consider them here.

10). The Vector Product of Two Vectors (Cross Product)

Having now extended into 3D, we can now consider the other form of vector multiplication, the so-called **vector product** of two vectors. This may seem a bit strange, but it does have practical applications.

For two vectors $\underline{v}_1 = (x_1 , y_1 , z_1)$ and $\underline{v}_2 = (x_2 , y_2 , z_2)$ we define the vector product as follows:

$$\underline{v}_1 \times \underline{v}_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} ,$$

where the right-hand-side is the determinant of a 3×3 matrix. Recall that this is determined using cofactors:

$$\text{cofactor of } \underline{i} = + \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}$$

$$\text{cofactor of } \underline{j} = - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}$$

$$\text{cofactor of } \underline{k} = + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} .$$

That means the vector product is given by

$$v_1 \times v_2 = \underline{i} \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} - \underline{j} \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} + \underline{k} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} .$$

Notes

(i). Unlike the dot product which takes two vectors and returns a number, the vector product returns another vector.

(ii). The resultant product vector will be perpendicular two both of the original vectors. See in class for the "right-hand rule".

(iii). The vector product is often called the **cross product**.

(iv). When writing a cross product on paper we can use $\wedge$ instead of $\times$; this avoids confusion with an *x*. It's still referred to as a cross product!

(v). Applications include moments of forces (torques), shearing forces and rotational dynamics.

Summary Examples in 3D

(6). For the 3-dimensional vectors $\underline{v}_1 = (2, -1, 3)$ and $\underline{v}_2 = (3, 2, 4)$ determine

(i). $\underline{v}_1 + \underline{v}_2$;

(ii). $\underline{v}_1 - \underline{v}_2$;

(iii). $\underline{v}_1 \cdot \underline{v}_2$;

(iv). $\underline{v}_1 \times \underline{v}_2$ $\quad$ (Alternative notation : $\underline{v}_1 \wedge \underline{v}_2$) ;

(v). The angle of separation between the vectors;

(vi). The component of $\underline{v}_1$ in the direction of $\underline{v}_2$.

(i). $\underline{v}_1 + \underline{v}_2 = (2+3, (-1)+2, 3+4) = \underline{\underline{(5, 1, 7)}}$

(ii). $\underline{v}_1 - \underline{v}_2 = (2-3, (-1)-2, 3-4) = \underline{\underline{(-1, -3, -1)}}$

(iii). $\underline{v}_1 \cdot \underline{v}_2 = 2\times 3 + (-1)\times 2 + 3\times 4 = \underline{\underline{16}}$

(iv). $$v_1 \times v_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & 3 \\ 3 & 2 & 4 \end{vmatrix}$$

$$= \underline{i}\begin{vmatrix} -1 & 3 \\ 2 & 4 \end{vmatrix} - \underline{j}\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} + \underline{k}\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}$$

$$= \underline{i}\,[(-1)\times 4 - 3\times 2] - \underline{j}\,[2\times 4 - 3\times 3] + \underline{k}\,[2\times 2 - (-1)\times 3]$$

$$= \underline{i}\,[-4 - 6] - \underline{j}\,[8 - 9] + \underline{k}\,[4 + 3]$$

$$= -10\,\underline{i} + 1\,\underline{j} + 7\,\underline{k}$$

$$= \underline{\underline{(-10, 1, 7)}}$$

(v). Over . . .

(v). $\underline{v}_1 \cdot \underline{v}_2 \; = \; 2 \times 3 \; + \; (-1) \times 2 \; + \; 3 \times 4 \; = \; 16$

$$|\, \underline{v}_1 \,| \; = \; \sqrt{2^2 + (-1)^2 + 3^2} \; = \; \sqrt{14}$$

$$|\, \underline{v}_2 \,| \; = \; \sqrt{3^2 + 2^2 + 4^2} \; = \; \sqrt{29} \; .$$

From the definition of the dot product

$$\begin{aligned} \cos\theta \; &= \; \frac{\underline{v}_1 \cdot \underline{v}_2}{|\,\underline{v}_1\,|\,|\,\underline{v}_2\,|} \\ &= \; \frac{16}{\sqrt{14}\,\sqrt{29}} \\ &= \; 0.79407 \end{aligned}$$

and so the angle of separation is $\underline{\underline{\theta \; = \; 37.43^\circ}}$.

(vi). Magnitude of $\underline{v}_2$: $\quad |\, \underline{v}_2 \,| \; = \; \sqrt{3^2 + 2^2 + 4^2} \; = \; \sqrt{29}$

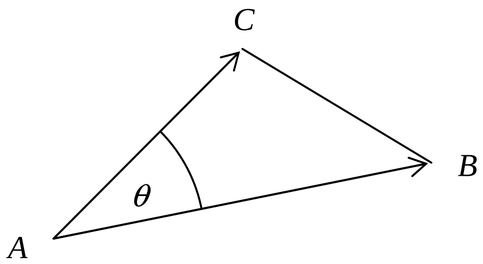
Unit direction vector: $\underline{u} \; = \; \dfrac{\underline{v}_2}{|\,\underline{v}_2\,|} \; = \; \dfrac{1}{\sqrt{29}}\,(3\,,\,2\,,\,4)$

Component of $\underline{v}_1$ in the direction of $\underline{v}_2$:

$$\begin{aligned} \underline{u} \cdot \underline{v}_1 \; &= \; \frac{1}{\sqrt{29}}\,(3\,,\,2\,,\,4) \cdot (2\,,\,-1\,,\,3) \\ &= \; \frac{1}{\sqrt{29}}\,(3\times2 \; + \; 2\times(-1) \; + \; 4\times3) \\ &= \; \frac{1}{\sqrt{29}} \times 16 \\ &= \; \underline{\underline{2.9711}} \end{aligned}$$

(7). A triangle in 3D space has vertices $A(3\,,2\,,4)$, $B(1\,,-4\,,5)$ and $C(2\,,0\,,-1)$. Determine the angle between the sides AB and AC.

Angle between sides AB and AC is the angle of separation of the relative position vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.



$\overrightarrow{AB} \;=\; (1-3\,,-4-2\,,5-4) \;=\; (-2\,,-6\,,1)$ $\qquad \left|\overrightarrow{AB}\right| \;=\; \sqrt{41}$

$\overrightarrow{AC} \;=\; (2-3\,,0-2\,,-1-4) \;=\; (-1\,,-2\,,-5)$ $\qquad \left|\overrightarrow{AC}\right| \;=\; \sqrt{30}$

$\overrightarrow{AB}\cdot\overrightarrow{AC} \;=\; (-2)\times(-1) \;+\; (-6)\times(-2) \;+\; 1\times(-5) \;=\; 9$

$$\cos\theta \;=\; \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|}$$

$$=\; \frac{9}{\sqrt{41}\,\sqrt{30}}$$

$$=\; 0.25662$$

Angle: $\qquad \theta \;=\; 75.13^\circ$

Tutorial Exercises

Two-Dimensional Vectors

Q1. Sketch the following vectors (given in polar form) emanating from the specified points:

(i). $(2\ ,\ 30°)$ from $(-3\ ,\ 2)$ **(ii).** $(5\ ,140°)$ from $(1\ ,\ 4)$

(iii). $(0.25\ ,\ 200°)$ from $(5\ ,\ 4)$ **(iv).** $(6\ ,\ -30°)$ from $(-2\ ,\ 2)$.

Q2. Sketch the following vectors (given in rectangular form) emanating from the specified points:

(i). $(3\ ,\ 4)$ from $(2\ ,\ 1)$ **(ii).** $(-4\ ,2)$ from $(2\ ,\ -4)$

(iii). $(6.5\ ,\ -3.5)$ from $(-4\ ,\ -4)$ **(iv).** $(-2.5\ ,\ -4.5)$ from $(-6\ ,\ 2)$.

Q3. Convert the following vectors to rectangular (Cartesian) form, first using the conversion formulae stated on p8, and then using the conversion functions on your calculator:

(i). $(2\ ,\ 30°)$ **(ii).** $(3\ ,\ 80°)$

(iii). $(1\ ,120°)$ **(iv).** $(5\ ,\ 315°)$

(v). $(4\ ,\ 200°)$ **(vi).** $(2\ ,\ -75°)$.

Q4. Convert the following vectors to rectangular (Cartesian) form without the aid of conversion formulae or calculator:

(i). $(4\ ,\ 0°)$ **(ii).** $(8\ ,\ 90°)$

(iii). $(6\ ,\ 180°)$ **(iv).** $(5\ ,\ 270°)$

(v). $(3\ ,\ -90°)$ **(vi).** $(10\ ,\ -180°)$.

Q5. Convert the following vectors to polar form, first using the conversion formulae stated on p8, and then using the conversion functions on your calculator:

(i). $(-3\ ,\ 2)$ **(ii).** $(3\ ,\ -2)$

(iii). $(-4\ ,\ -3)$ **(iv).** $(4\ ,\ 3)$

(v). $(-2\ ,\ 1)$ **(vi).** $(2\ ,\ -1)$.

Q6. Convert the following vectors to polar form without the aid of conversion formulae or calculator:

(i). $(5\,,\,0)$ **(ii).** $(0\,,\,10)$

(iii). $(-8\,,\,0)$ **(iv).** $(0\,,\,-2)$.

Q7. Given the vectors $\underline{v}_1 = (4\,,\,-3)$, $\underline{v}_2 = (-2\,,\,4)$ and $\underline{v}_3 = (-5\,,\,-1)$, determine the following, displaying the results graphically:

(i). $\underline{v}_1 + \underline{v}_2$ **(ii).** $\underline{v}_1 + \underline{v}_2 + \underline{v}_3$

(iii). $\underline{v}_1 + (-\underline{v}_2)$ **(iv).** $\underline{v}_1 - \underline{v}_3$

(v). $2\underline{v}_1$ **(vi).** $3\underline{v}_2$

(vii). $2\underline{v}_1 + 3\underline{v}_2$ **(viii).** $4\underline{v}_1 - 2\underline{v}_2 + 5\underline{v}_3$.

Q8. For each of the following pairs of vectors, determine the magnitudes $|\underline{v}_1|$, $|\underline{v}_2|$, the scalar (dot) product $\underline{v}_1 \cdot \underline{v}_2$ and the angle of separation θ :

(i). $\left.\begin{array}{l} \underline{v}_1 = (2\,,\,2) \\ \underline{v}_2 = (1\,,\,-3) \end{array}\right\}$ **(ii).** $\left.\begin{array}{l} \underline{v}_1 = (-4\,,\,6) \\ \underline{v}_2 = (5\,,\,-8) \end{array}\right\}$

(iii). $\left.\begin{array}{l} \underline{v}_1 = (3\,,\,4) \\ \underline{v}_2 = (1\,,\,0) \end{array}\right\}$ **(iv).** $\left.\begin{array}{l} \underline{v}_1 = (3\,,\,4) \\ \underline{v}_2 = (0\,,\,1) \end{array}\right\}$.

Q9. A triangle has vertices $A(1\,,\,1)$, $B(4\,,\,2)$ and $C(3\,,\,4)$.

(i). Draw this triangle accurately on an *Oxy* axes system.

(ii). Determine the relative position vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

(iii). **By calculation**, determine the angle of the triangle at vertex A .

(iv). Confirm your answer by the direct measurement of the angle on your diagram.

Q10. A triangle has vertices $A(-2\,,1)$, $B(3\,,1)$ and $C(1\,,5)$.

(i). Draw this triangle accurately on an *Oxy* axes system.

(ii). **By calculation**, determine the angle of the triangle at vertex B.

(iii). Confirm your answer by the direct measurement of the angle on your diagram.

Q11. For each of the following pairs of vectors, determine the component of $\underline{v}_1$ in the direction of $\underline{v}_2$ (see p13 and the 3D Example (6)(vi) on pp16-17):

(i). $\left.\begin{aligned} \underline{v}_1 &= (2\,,2) \\ \underline{v}_2 &= (1\,,-3) \end{aligned}\right\}$ **(ii).** $\left.\begin{aligned} \underline{v}_1 &= (-4\,,6) \\ \underline{v}_2 &= (5\,,-8) \end{aligned}\right\}$.

Three-Dimensional Vectors

Q12. Given the vectors $\underline{v}_1 = (4\,,-5\,,1)$, $\underline{v}_2 = (5\,,6\,,-3)$ and $\underline{v}_3 = (6\,,2\,,-4)$, determine the following :

(i). $\underline{v}_1 + \underline{v}_2 + \underline{v}_3$

(ii). $4\underline{v}_1 + 3\underline{v}_2 - 2\underline{v}_3$

(iii). $\underline{v}_1 + (-\underline{v}_3)$

(iv). $-2\underline{v}_1 + \underline{v}_2 + 5\underline{v}_3$

(v). $\underline{v}_1 \cdot \underline{v}_2$

(vi). $\underline{v}_2 \cdot \underline{v}_3$

(vii). $\underline{v}_1 \times \underline{v}_2$ (Alternative notation : $\underline{v}_1 \wedge \underline{v}_2$)

(viii). $\underline{v}_2 \times \underline{v}_3$ and $\underline{v}_3 \times \underline{v}_2$ [Comment?]

(ix). The angle of separation between $\underline{v}_1$ and $\underline{v}_2$.

(x). The angle of separation between $\underline{v}_1$ and $\underline{v}_3$.

(xi). The component of $\underline{v}_1$ in the direction of $\underline{v}_2$.

(xii). The component of $\underline{v}_3$ in the direction of $\underline{v}_1$.

Q13. A triangle in 3D space has vertices $A(1, 4, 3)$, $B(4, 2, 0)$ and $C(5, 4, 6)$.

(i). Determine the relative position vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

(ii). Determine the angle of the triangle at vertex A .

Q14. A triangle in 3D space has vertices $A(-2, 3, -5)$, $B(2, 0, 4)$ and $C(1, 5, 1)$. Determine the angle of the triangle at vertex C .

Q15. The unit vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$ are written in Cartesian form as follows:

$$\underline{i} = (1, 0, 0)$$
$$\underline{j} = (0, 1, 0)$$
$$\underline{k} = (0, 0, 1) \ .$$

(i). Determine the scalar (dot) products $\underline{i} \cdot \underline{i}$, $\underline{j} \cdot \underline{j}$, $\underline{k} \cdot \underline{k}$, $\underline{i} \cdot \underline{j}$, $\underline{i} \cdot \underline{k}$ and $\underline{j} \cdot \underline{k}$.

(ii). Determine the vector (cross) products $\underline{i} \times \underline{i}$, $\underline{j} \times \underline{j}$, $\underline{k} \times \underline{k}$, $\underline{i} \times \underline{j}$, $\underline{j} \times \underline{k}$ and $\underline{k} \times \underline{i}$.

Q16. A force $\underline{F} = (3, -2, 5)$ acts on a particle which moves from point $P(1, 4, -1)$ to point $Q(-2, 3, 1)$.

(i). Determine the displacement vector $\underline{d}$ of the particle.

(ii). Determine the work done by the force, i.e. $\underline{F} \cdot \underline{d}$.

Q17. A force $\underline{F}$ acts through a point P . The moment of force (torque) about a second point Q is given by

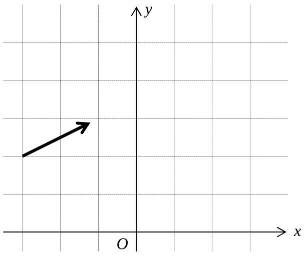
$$\underline{M} = \underline{r} \times \underline{F} \ ,$$

where $\underline{r}$ is the position vector of Q relative to P . Determine $\underline{M}$ when

$$F = 4\,\underline{i} + 5\,\underline{j} - 2\,\underline{k}$$
$$P \text{ is } (2, 3, -5)$$
$$Q \text{ is } (1, 2, -3) \ .$$

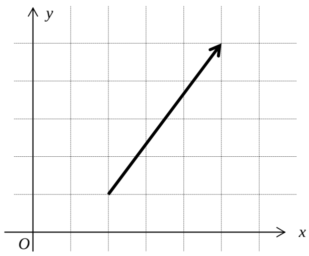
Answers to Tutorial Exercises

A1. **(i).**



(ii). – (iv). Similar

A2. **(i).**



(ii). – (iv). Similar

A3. Figures, where approximated, expressed to 3 decimal places.

(i). $(1.732\ ,\ 1)$ **(ii).** $(0.521\ ,\ 2.954)$

(iii). $(-0.5\ ,\ 0.866)$ **(iv).** $(3.536\ ,\ -3.536)$

(v). $(-3.759\ ,\ -1.368)$ **(vi).** $(0.518\ ,\ -1.932)$

A4. Hint: Sketch the vector first.

(i). $(4\ ,\ 0)$ **(ii).** $(0\ ,\ 8)$

(iii). $(-6\ ,\ 0)$ **(iv).** $(0\ ,\ -5)$

(v). $(0\ ,\ -3)$ **(vi).** $(-10\ ,\ 0)$

A5. Figures, where approximated, expressed to 3 decimal places.

(i). $(3.606\ ,\ 146.310^\circ)$ **(ii).** $(3.606\ ,\ -33.690^\circ)$

(iii). $(5\ ,\ -143.130^\circ)$ **(iv).** $(5\ ,\ 36.870^\circ)$

(v). $(2.236\ ,\ 153.435^\circ)$ **(vi).** $(2.236\ ,\ -26.565^\circ)$

A6. Hint: Sketch the vector first.

(i). $(5\ ,\ 0°)$ **(ii).** $(10\ ,\ 90°)$

(iii). $(8\ ,\ 180°)$ **(iv).** $(2\ ,\ -90°)$ or $(2\ ,\ 270°)$

A7. **(i).** $\underline{v}_1 + \underline{v}_2 = (2\ ,\ 1)$ **(ii).** $\underline{v}_1 + \underline{v}_2 + \underline{v}_3 = (-3\ ,\ 0)$

(iii). $\underline{v}_1 + (-\underline{v}_2) = (6\ ,\ -7)$ **(iv).** $\underline{v}_1 - \underline{v}_3 = (9\ ,\ -2)$

(v). $2\underline{v}_1 = (8\ ,\ -6)$ **(vi).** $3\underline{v}_2 = (-6\ ,\ 12)$

(vii). $2\underline{v}_1 + 3\underline{v}_2 = (2\ ,\ 6)$ **(viii).** $4\underline{v}_1 - 2\underline{v}_2 + 5\underline{v}_3 = (-5\ ,\ -25)$

A8. **(i).** $|\underline{v}_1| = \sqrt{8}$ $|\underline{v}_2| = \sqrt{10}$ $\underline{v}_1 \cdot \underline{v}_2 = -4$ $\theta = 116.57°$

(ii). $|\underline{v}_1| = \sqrt{52}$ $|\underline{v}_2| = \sqrt{89}$ $\underline{v}_1 \cdot \underline{v}_2 = -68$ $\theta = 178.32°$

(iii). $|\underline{v}_1| = 5$ $|\underline{v}_2| = 1$ $\underline{v}_1 \cdot \underline{v}_2 = 3$ $\theta = 53.13°$

(iv). $|\underline{v}_1| = 5$ $|\underline{v}_2| = 1$ $\underline{v}_1 \cdot \underline{v}_2 = 4$ $\theta = 36.87°$

A9. **(ii).** $\overrightarrow{AB} = (3\ ,\ 1)$ $\overrightarrow{AC} = (2\ ,\ 3)$ **(iii).** $\theta = 37.9°$

A10. **(ii).** $\theta = 63.4°$

A11. **(i).** $\dfrac{-4}{\sqrt{10}} \approx -1.265$ **(ii).** $\dfrac{-68}{\sqrt{89}} \approx -7.208$

A12. **(i).** $(15\ ,\ 3\ ,\ -6)$ **(ii).** $(19\ ,\ -6\ ,\ 3)$

(iii). $(-2\ ,\ -7\ ,\ 5)$ **(iv).** $(27\ ,\ 26\ ,\ -25)$

(v). -13 **(vi).** 54

(vii). $(9\ ,\ 17\ ,\ 49)$ **(viii).** $(-18\ ,\ 2\ ,\ -26)$; $(18\ ,\ -2\ ,\ 26)$

(ix). 103.87° **(x).** 78.1°

(xi). -3.153 **(xii).** 1.543

A13. (ii). $\overrightarrow{AB} = (3\,,-2\,,-3)$ $\quad \overrightarrow{AC} = (4\,,0\,,3)$ $\quad$ (iii) $\theta = 82.65°$

A14. 105.40°

A15. (i). $\underline{i} \bullet \underline{i} = \underline{j} \bullet \underline{j} = \underline{k} \bullet \underline{k} = 1$

$\underline{i} \bullet \underline{j} = \underline{i} \bullet \underline{k} = \underline{j} \bullet \underline{k} = 0$

(ii). $\underline{i} \times \underline{i} = \underline{i} \times \underline{i} = \underline{i} \times \underline{i} = (0\,,0\,,0) = \underline{0}$ (i.e. the zero vector)

$\underline{i} \times \underline{j} = \underline{k} \quad ; \quad \underline{j} \times \underline{k} = \underline{i} \quad ; \quad \underline{k} \times \underline{i} = \underline{j}$

A16. (i). $\underline{d} = \overrightarrow{PQ} = (-3\,,-1\,,2)$

(ii). $\underline{F} \bullet \underline{d} = 3$

A17. $\underline{r} = \overrightarrow{PQ} = (-1\,,-1\,,2)$

$\underline{M} = \underline{r} \times \underline{F} = (-8\,,6\,,-1) = -8\,\underline{i} + 6\,\underline{j} - \underline{k}$

Note: A torque produces a rotation. The direction of the torque vector gives the direction of the axis of that rotation.