# School of Computing, Engineering & Built Environment

# Mathematics Summer School

## Level 2 Entry – Engineering

## Differential Calculus

# Contents

# Differential Calculus

## (1). Review of Function Notation

A function is essentially a mathematical rule for determining the value of one quantity (the **dependent variable**) from the value of another (the **independent variable**).

**Example**

**(1)**. $y = x^2$

$x$ - independent variable

$y$ - dependent variable.

We can refer to the LHS as $f(x)$ (read as "$f$ of $x$"):

$$f(x) = x^2 \ .$$

Using this notation, we can refer to specific values of the function:

$$f(x) = x^2$$

$$f(2) = 2^2 = 4$$

$$f(4.5) = 4.5^2 = 20.25$$

$$f(x + \Delta x) = (x + \Delta x)^2 = x^2 + 2x\Delta x + \Delta x^2 \ .$$

By plotting $f(x)$ against $x$ we can determine the graph of the function:

| $x$ | –2 | –1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | 4 | 1 | 0 | 1 | 4 |

See **Figure 1** over the page.

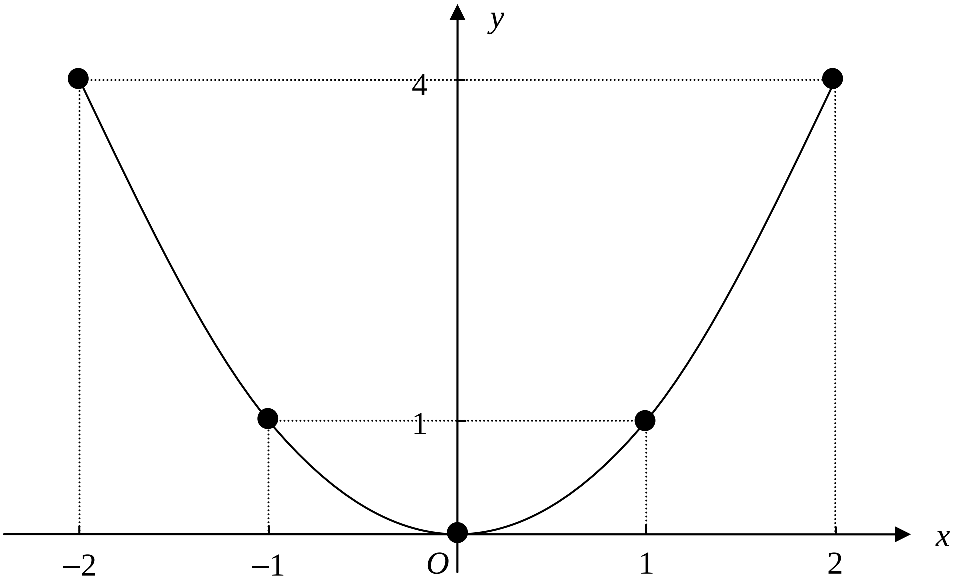
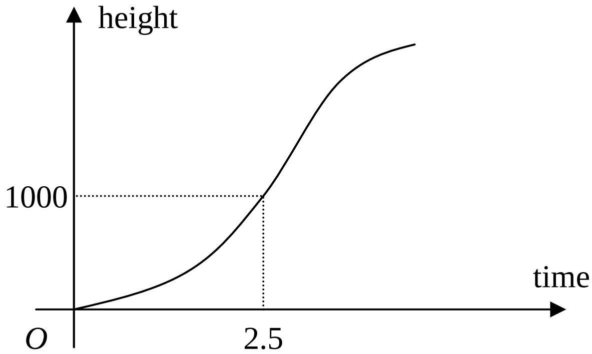



**Figure 1**

## (2). Average Rate of Change of a Function

Consider a projectile fired into the air. Suppose this projectile rose to a height of 1000m in a time of 2.5s:




$$\frac{1000}{2.5} = 400$$

**Figure 2**

We say it travelled at an **average** vertical velocity of 400 $ms^{-1}$. This is an example of an **average rate of change** of one variable with respect to another:

$$\text{Average vertical velocity} = \frac{\text{Change in height}\,(\Delta h)}{\text{Change in time}\,(\Delta t)}\ .$$

For a general function $y = f(x)$,

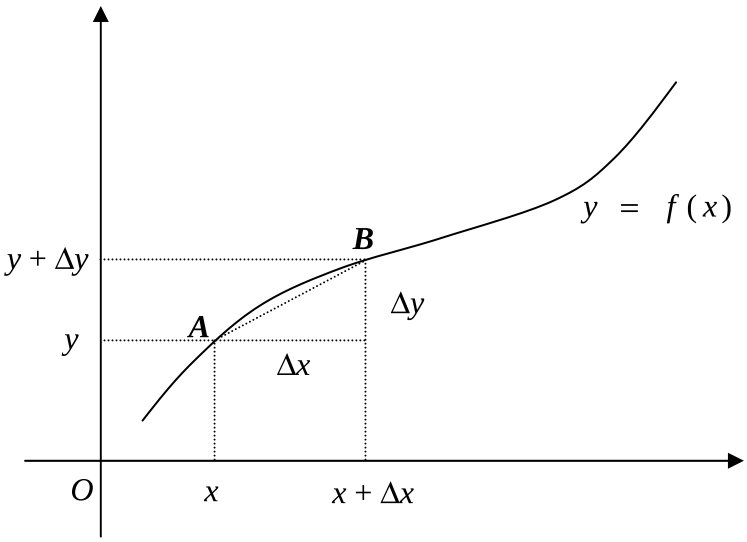



**Figure 3**

the average rate of change of *y* with respect to *x* as *x* changes by an amount $\Delta x$ is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} .$$

**Example**

**(2).** For the function

$$y = x^2 - 6x + 5$$

**(i).** determine the average rate of change of $y$ per unit change in $x$ ;

**(ii).** determine the average rate of change of the function as $x$ changes from $4$ to $7$ .

**Solution**

**(i).**
$$f(x) = x^2 - 6x + 5$$

$$\begin{aligned} f(x + \Delta x) &= (x + \Delta x)^2 - 6(x + \Delta x) + 5 \\ &= x^2 + 2x.\Delta x + \Delta x^2 - 6x - 6\Delta x + 5 \\ &= (x^2 - 6x + 5) + (2x - 6)\Delta x + \Delta x^2 \end{aligned}$$

Now form /

Now form the average rate of change:

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{(2x - 6)\,\Delta x + \Delta x^2}{\Delta x}$$

$$= \underline{\underline{(2x - 6) + \Delta x}}$$

**(ii).** As $x$ changes from $4$ to $7$: $x = 4$ , $\Delta x = 3$

$$\underline{\underline{\frac{\Delta y}{\Delta x} = (2\times 4 - 6) + 3 = 5}}$$

**(3). Instantaneous Rate of Change**

In the example of the projectile, we arrived at a single value of 400 $ms^{-1}$ for its average vertical velocity over the first 2.5s of its flight. This figure tells us nothing of the vertical velocity at any **instant** of time. For example, what can we say about its vertical velocity exactly 2.5s into the flight? For a single point in time we have neither a "change in height", nor a "change in time" to divide by. What we could do is look at the average rate of change close to the point of interest over smaller and smaller intervals of time.

Consider the general case of $y = f(x)$ again, depicted in **Figure 3** above. The **instantaneous** rate of change of $y$ with respect to $x$ is found by letting $\Delta x$ become smaller and smaller, and noting what happens to the ratio

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad .$$

Mathematically, we are looking for the **limiting value** of the average rate of change as $\Delta x$ tends towards zero. We write it as

$$\text{Instantaneous rate of change} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) \quad .$$

We call this the **derivative** of $y$ with respect to $x$ and use the following notation:

$$y = f(x)$$

$$\frac{dy}{dx} = f'(x),$$

read as "d $y$ by d $x$ equals $f$ dashed (or $f$ prime) of $x$".

Graphically, we have:

$$\frac{\Delta y}{\Delta x} \; = \; \text{gradient of the straight line } AB \;\; ;$$

$$\frac{dy}{dx} \; = \; \text{gradient of the tangent to the curve at } A \;\; .$$

**Example**

**(3).** Determine the derivative of $y = x^2 - 6x + 5$ from "first principles".

From Example (2) (i) the average rate of change is:

$$\frac{\Delta y}{\Delta x} = (2x - 6) + \Delta x \; .$$

Let $\Delta x$ get smaller and smaller (i.e. tend to zero). All that we are left with is $2x - 6$, so

$$\underline{\underline{\frac{dy}{dx} = 2x - 6}} \; .$$

The value of the derivative at any particular point tells us whether the function is increasing, decreasing or doing neither:

$$\frac{dy}{dx} > 0 \quad \rightarrow \quad \text{Increasing}$$

$$\frac{dy}{dx} < 0 \quad \rightarrow \quad \text{Decreasing}$$

$$\frac{dy}{dx} = 0 \quad \rightarrow \quad \text{Neither (stationary point)} \; .$$

We shall use this information later for curve sketching.

**Example**

**(4).** For the function $y = x^2 + 1$ :

**(i).** graph $y$ against $x$ for $-2 \leq x \leq +2$;

**(ii).** determine $\frac{\Delta y}{\Delta x}$;

**(iii).** determine $\frac{dy}{dx}$

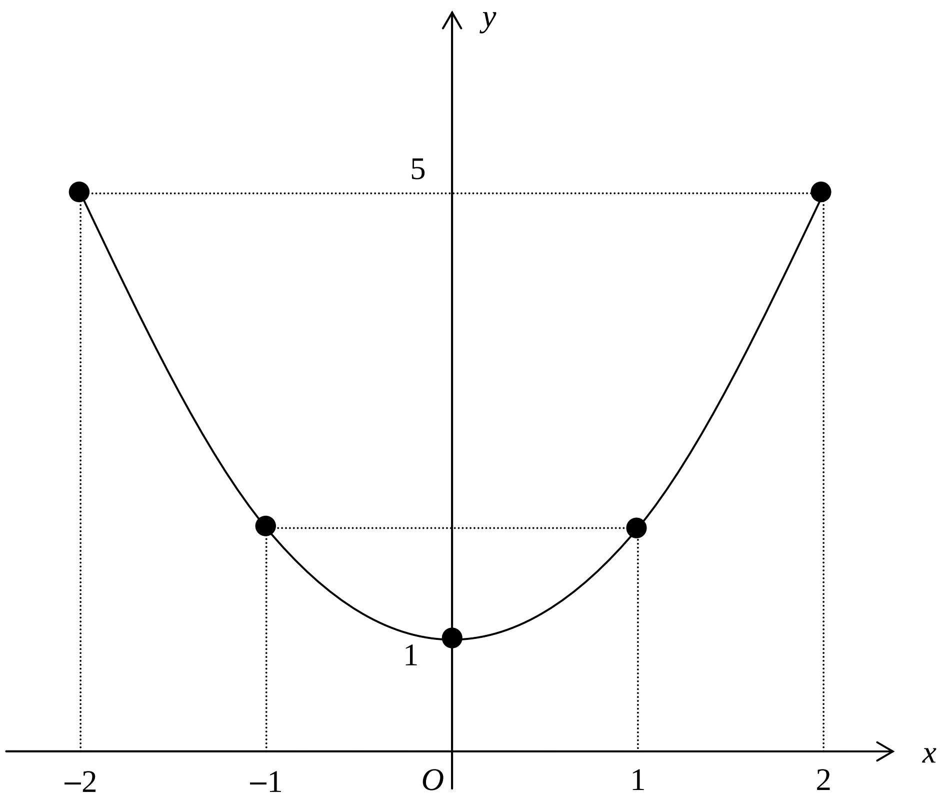
**(iv).** evaluate $\frac{dy}{dx}$ for $x = -2, -1, 0, +1, +2$;

**(v).** relate the value of the derivative to the behaviour of the graph.

**Solution**

**(i).** $y = x^2 + 1$

| $x$ | $-2$ | $-1$ | $0$ | $+1$ | $+2$ |
|---|---|---|---|---|---|
| $y$ | 5 | 2 | 1 | 2 | 5 |




**(ii).** /

**(ii).** $$f(x) = x^2 + 1$$

$$\begin{aligned} f(x + \Delta x) &= (x + \Delta x)^2 + 1 \\ &= x^2 + 2x.\Delta x + \Delta x^2 + 1 \\ &= (x^2 + 1) + 2x.\Delta x + \Delta x^2 \end{aligned}$$

Now form the average rate of change:

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \frac{2x.\Delta x + \Delta x^2}{\Delta x} \\ &= \underline{\underline{2x + \Delta x}} \end{aligned}$$

**(iii).** Derivative:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \left[ 2x + \Delta x \right] = \underline{\underline{2x}}$$

**(iv). / (v).**

| $x$ | –2 | –1 | 0 | +1 | +2 |
|---|---|---|---|---|---|
| $\frac{dy}{dx}$ | –4 | –2 | 0 | +2 | +4 |
| Graph | ↘ | ↘ | → | ↗ | ↗ |

## (4). Derivatives of Polynomials and Other Functions

To work out derivatives we rarely go through the limiting process. Certain patterns emerge from which rules can be derived.

Below is a table of powers of *x*:

| $f(x)$ | $f'(x)$ |
|---|---|
| $1 = x^0$ | $0$ |
| $x = x^1$ | $1$ |
| $x^2$ | $2x$ |
| $x^3$ | $3x^2$ |
| $x^4$ | $4x^3$ |
| . | . |
| $x^n$ | $nx^{n-1}$ |

From this we have a rule for differentiating powers of *x*, which can be shown to hold for any power, whole number or otherwise.

Using the notation

$$\frac{d}{dx}[\quad]$$

to mean "differentiate the contents of the brackets with respect to *x*", it is readily shown that

$$\frac{d}{dx}[\, a\, f(x)\,] = a\, f'(x) \;,\; \text{where } a \text{ is a constant,}$$

and

$$\frac{d}{dx}[\, f(x) + g(x)\,] = f'(x) + g'(x) \;.$$

This means that we can differentiate a string of terms term-by-term.

**Examples**

**(5).** $y = 2x^2 - 5x + 4$

$$\frac{dy}{dx} = 2(2x) - 5(1) + 0$$
$$= \underline{\underline{4x - 5}}$$

**(6).** $y = \dfrac{x^2 + 2x - 6}{x}$

Note that we cannot just differentiate top and bottom separately; we first divide out:

$$y = x + 2 - \frac{6}{x}$$
$$= x + 2 - 6x^{-1}$$

$$\frac{dy}{dx} = 1 + 0 - 6(-x^{-2})$$
$$= 1 + 6x^{-2}$$
$$= \underline{\underline{1 + \frac{6}{x^2}}}$$

**(7).** $y = \sqrt{x} = x^{1/2}$

$$\frac{dy}{dx} = \tfrac{1}{2}x^{-1/2}$$
$$= \frac{1}{2x^{1/2}}$$
$$= \underline{\underline{\frac{1}{2\sqrt{x}}}}$$

**(8).** The vertical displacement of a projectile is given as a function of time by

$$s = -2t^2 + 3t - 4 .$$

Determine the projectile's vertical velocity and acceleration for any value of $t$ .

Note: $v = \frac{ds}{dt}$ and $a = \frac{dv}{dt}$ .

$$v = \frac{ds}{dt} = \underline{\underline{-4t + 3}}$$

$$a = \frac{dv}{dt} = \underline{\underline{-4}}$$

The derivatives of some "standard" functions are stated below without proof :

| $f(x)$ | $f'(x)$ |
|---|---|
| $\sin(x)$ | $\cos(x)$ |
| $\cos(x)$ | $-\sin(x)$ |
| $\tan(x)$ | $\sec^2(x)$ |
| $e^x$ | $e^x$ |
| $\ln(x)$ | $\frac{1}{x}$ |

Note that in calculus all "trig angles" must be in **radians** and not in degrees.

Calculus and degrees do not mix well!

This table contains functions of $x$ differentiated with respect to $x$. Later, you will have to use this table as if it were expressed in terms of other variables, i.e. functions of $u$ differentiated with respect to $u$, functions of $v$ differentiated with respect to $v$, etc..

## (5). Alternative Notation

Sometimes we use the following shorthand notation:

$$y' = \frac{dy}{dx} \qquad\qquad \dot{x} = \frac{dx}{dt} \quad .$$

## (6). Electrical Applications of the Derivative

In electronic circuits where voltages and currents are changing with repect to one another, certain relationships can be expressed in terms of derivatives. Below are two such cases:

Law of Capacitance: $i = C \dfrac{dV}{dt}$

Law of Inductance: $V = L \dfrac{di}{dt}$ .

## (7). Derivatives of Products and Quotients

We now begin looking at how to differentiate more complicated functions where "first principles" is not really an option. The following rules of differentiation are extremely important.

### (a). The Product Rule

If

$$y = f(x)\, g(x)$$

then

$$\frac{dy}{dx} = f'(x)\, g(x) + f(x)\, g'(x) \quad .$$

**IMPORTANT: Note that** $y' \neq f'(x)\, g'(x)$ **.**

**Example**/

**Example**

**(9).** Determine $\frac{dy}{dx}$ when $y = (x^2 - 1) \sin x$ .

$$f(x) = x^2 - 1 \qquad\qquad g(x) = \sin x$$

$$f'(x) = 2x \qquad\qquad g'(x) = \cos x$$

$$\begin{aligned} \frac{dy}{dx} &= f'(x)\, g(x) + f(x)\, g'(x) \\ &= \underline{\underline{2x \sin x + (x^2 - 1) \cos x}} \ . \end{aligned}$$

**(b). The Quotient Rule**

If

$$y = \frac{f(x)}{g(x)}$$

then

$$\frac{dy}{dx} = \frac{f'(x)\, g(x) - f(x)\, g'(x)}{[\, g(x)\, ]^2} \ .$$

**IMPORTANT: Note that** $y' \neq \frac{f'(x)}{g'(x)}$ **.**

**Example**

**(10).** Determine $\frac{dy}{dx}$ when $y = \frac{x^3 - 1}{x^2 + 1}$ .

$$f(x) = x^3 - 1 \qquad\qquad g(x) = x^2 + 1$$

$$f'(x) = 3x^2 \qquad\qquad g'(x) = 2x$$

Now input these expressions into the quotient rule formula:

$$\frac{dy}{dx} = \frac{f'(x)\, g(x) \; - \; f(x)\, g'(x)}{[\, g(x)\, ]^2}$$

$$= \frac{3x^2\, (x^2+1) \; - \; (x^3-1)\, 2x}{(x^2+1)^2}$$

$$= \frac{(3x^4+3x^2) \; - \; (2x^4-2x)}{(x^2+1)^2}$$

$$= \frac{x^4 + 3x^2 + 2x}{(x^2+1)^2}$$

$$= \underline{\underline{\frac{x(x^3+3x+2)}{(x^2+1)^2}}} \; .$$

**(8). The Chain Rule**

Now we shall look at what is probably the most powerful rule of differentiation, the Chain Rule, and its "wee brother" the Power Rule.

**Examples**

**(11). (i).** Suppose we have the function

$$y = (4x-3)^{75} \; .$$

How do we determine the derivative? Expanding the bracket isn't practical. Instead we use the **power rule**:

$$y = [\, g(x)\, ]^n$$

$$\frac{dy}{dx} = n\, [\, g(x)\, ]^{n-1}\, g'(x) \; .$$

We differentiate the square bracket as if it were a simple power, then multiply by the derivative of the contents of the square bracket.

For the above example:

$$y = (4x-3)^{75}$$

$$y' = 75\, (4x-3)^{74} .4$$

$$= \underline{\underline{300\, (4x-3)^{74}}} \; .$$

**(ii).** $f(x) = (11x^2 - 6)^8$

$$f'(x) = 8(11x^2 - 6)^7 \, . \, 22x$$
$$= \underline{\underline{176x(11x^2 - 6)^7}}$$

The **Power Rule** is a special case of the more general **Chain Rule**.

Let $y = f(u)$ where $u = g(x)$ . Then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad .$$

**Examples**

**(12).** Repeating the earlier example where $y = (4x - 3)^{75}$:

Express this as

$$y = u^{75} \qquad \text{where} \qquad u = 4x - 3 \quad .$$

Differentiate:

$$\frac{dy}{du} = 75u^{74} \qquad \frac{du}{dx} = 4 \quad .$$

Invoke the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= 75u^{74} \, . \, 4$$

$$= 300u^{74}$$

$$= \underline{\underline{300(4x - 3)^{74}}} \quad .$$

**(13).** Determine $\frac{dy}{dx}$ when $y = \sin(2x)$.

Express this as: $y = \sin(u)$ where $u = 2x$ .

Differentiate:

$$\frac{dy}{du} = \cos(u) \qquad \frac{du}{dx} = 2 \ .$$

Chain rule:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= \cos(u) \,.\, 2$$

$$= 2\cos(u)$$

$$= \underline{\underline{2\cos(2x)}} \ .$$

**Note:** This can be generalised to give the result

$$\underline{\underline{y = \sin(ax) \quad \rightarrow \quad \frac{dy}{dx} = a\cos(ax)}} \quad , \quad \text{where } a \text{ is a constant.}$$

**Further Examples**

**(14).** $y = \cos(4x + 5)$

Write as: $y = \cos(u)$ $u = 4x + 5$

$$\frac{dy}{du} = -\sin(u) \qquad \frac{du}{dx} = 4$$

Chain rule:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= -\sin(u) \,.\, 4$$

$$= -4\sin(u)$$

$$= \underline{\underline{-4\sin(4x + 5)}} \ .$$

**(15).** $y = \ln(1 + 2x^3)$

Write as: $y = \ln u$ $\quad u = 1 + 2x^3$

$$\frac{dy}{du} = \frac{1}{u} = \frac{1}{1 + 2x^3} \qquad \frac{du}{dx} = 6x^2$$

Chain rule:

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

$$= \frac{1}{1 + 2x^3} \,.\, 6x^2$$

$$= \frac{6x^2}{1 + 2x^3} \,.$$

Applying the Chain Rule to "linear variants" of the standard functions and noting the patterns in the outcomes gives us an extented set of results that we can apply without the need for all the detail. [An extra useful result is included at the end of the table]:

| $f(x)$ | $f'(x)$ |
|---|---|
| $\sin(ax + b)$ | $a\cos(ax + b)$ |
| $\cos(ax + b)$ | $-a\sin(ax + b)$ |
| $\tan(ax + b)$ | $a\sec^2(ax + b)$ |
| $e^{ax + b}$ | $a\,e^{ax + b}$ |
| $\ln(ax + b)$ | $\frac{a}{ax + b}$ |
| $\ln(\,f(x)\,)$ | $\frac{f'(x)}{f(x)}$ |

## (9). Implicit Differentiation

A function of the form

$$y = f(x)$$

is said to express $y$ **explicitly** in terms of $x$. An expression like

$$x^2 + y^3 - x^3 y^2 = 1$$

gives $y$ **implicitly** in terms of $x$. We do not need $y$ explicitly in terms of $x$ to find $\frac{dy}{dx}$.

What we do is differentiate both sides of the equation with respect to $x$, using the chain rule to differentiate terms involving $y$.

For any function of $y$ we have

$$\frac{d}{dx}[\, g(y) \,] = \frac{d}{dy}[\, g(y) \,] \underline{\underline{\frac{dy}{dx}}} \; .$$

This means we differentiate a $y$-term in the way we might expect, but we need to multiply by the derivative of $y$ to ensure that the differentiation is with respect to $x$ and not $y$.

As an example,

$$\frac{d}{dx}[\, y^2 \,] = \frac{d}{dy}[\, y^2 \,] \frac{dy}{dx} = 2y \frac{dy}{dx} \; .$$

Once an implicit equation has been differentiated we solve for $\frac{dy}{dx}$.

**Example**

**(16).** Determine $\frac{dy}{dx}$ when $x^2 + y^3 - x^3 y^2 = 1$.

Differentiate both sides of the equation:

Requires the Product Rule

$$\frac{d}{dx}\left[\, x^2 + y^3 - x^3 y^2 \,\right] = \frac{d}{dx}[\, 1 \,]$$

$$2x + 3y^2 \frac{dy}{dx} - \left( 3x^2 y^2 + x^3 . 2y \frac{dy}{dx} \right) = 0$$

Note the derivatives

Open out the brackets and solve for $\frac{dy}{dx}$ :

$$2x \;+\; 3y^2 \frac{dy}{dx} \;-\; 3x^2 y^2 \;-\; x^3 .2y \frac{dy}{dx} \;=\; 0$$

$$3y^2 \frac{dy}{dx} \;-\; 2x^3 y \frac{dy}{dx} \;=\; 3x^2 y^2 \;-\; 2x$$

$$\left( 3y^2 \;-\; 2x^3 y \right) \frac{dy}{dx} \;=\; 3x^2 y^2 \;-\; 2x$$

$$\underline{\underline{\frac{dy}{dx} \;=\; \frac{(3x^2 y^2 \;-\; 2x)}{(3y^2 \;-\; 2x^3 y)}}} \;.$$

**(10). Logarithmic Differentiation**

Logarithmic differentiation is a special case of implicit differentiation and is useful when we have a function with a variable power or a function that contains multiple products and/or quotients.

First recall some results from logarithms:

**(i).** $\ln(x \,.\, y) \;=\; \ln(x) \;+\; \ln(y)$

**(ii).** $\ln\left( \frac{x}{y} \right) \;=\; \ln(x) \;-\; \ln(y)$

**(iii).** $\ln(x^p) \;=\; p \ln(x)$

**(iv).** $\ln(1) \;=\; 0$

**(v).** $\ln(e) \;=\; 1$ ,

and two from differentiation:

$$\frac{d}{dx}[\,\ln(y)\,] \;=\; \frac{1}{y}\frac{dy}{dx} \qquad \text{and} \qquad \frac{d}{dx}\ln[\,f(x)] \;=\; \frac{f'(x)}{f(x)} \;.$$

We are now ready to do logarithmic differentiation. The idea is that we can sometimes simplify a function before differentiating, by first taking natural logs. The process is best illustrated by a couple of examples.

**Examples**

**(17).** $y = 4^{x+1}$

Take natural logs of both sides:

$$\ln(y) = \ln(4^{x+1}) \ .$$

Now apply logarithm result (iii) stated earlier to bring down the power:

$$\ln(y) = (x+1)\ln(4)$$

$$\ln(y) = x\ln(4) + \ln(4) \ .$$

Note that $\ln(4)$ is just a number. Now we differentiate both sides of the equation using the result from implicit differentiation:

$$\frac{d}{dx}\ln(y) = \ln(4)\frac{d}{dx}x + \frac{d}{dx}\ln(4)$$

$$\frac{1}{y}\frac{dy}{dx} = \ln(4).1 + 0$$

$$\frac{1}{y}\frac{dy}{dx} = \ln(4)$$

$$\underline{\underline{\frac{dy}{dx} = y\ln(4)}} \ .$$

**(18).** $y = \dfrac{e^{3x}\sin(2x)}{\sqrt{1+x^2}}$

Take logs of both sides and sort out the product and the quotient using the first two log results stated earlier:

$$\ln(y) = \ln\left(\frac{e^{3x}\sin(2x)}{\sqrt{1+x^2}}\right)$$

$$\ln(y) = \ln[e^{3x}] + \ln[\sin(2x)] - \ln[\sqrt{1+x^2}]$$

Recall that a square root can be expressed as a power of a half:

$$\ln(y) = \ln[e^{3x}] + \ln[\sin(2x)] - \ln[(1+x^2)^{1/2}]$$

Now use the 3rd log result to deal with the powers:

$$\ln(y) \;=\; 3x\;\ln[e] \;+\; \ln[\sin(2x)] \;-\; \tfrac{1}{2}\ln[(1+x^2)]$$

Remember $\ln[e] \;=\; 1$:

$$\ln(y) \;=\; 3x \;+\; \ln[\sin(2x)] \;-\; \tfrac{1}{2}\ln[(1+x^2)]$$

Apply implicit differentiation:

$$\frac{1}{y}\frac{dy}{dx} \;=\; 3 \;+\; \frac{2\cos(2x)}{\sin(2x)} \;-\; \frac{1}{2}\,\frac{2x}{1+x^2}$$

$$\frac{1}{y}\frac{dy}{dx} \;=\; 3 \;+\; \frac{2\cos(2x)}{\sin(2x)} \;-\; \frac{x}{1+x^2}$$

$$\frac{dy}{dx} \;=\; y\left[\,3 \;+\; \frac{2\cos(2x)}{\sin(2x)} \;-\; \frac{x}{1+x^2}\,\right] .$$

It can be left like this or expressed entirely in terms of $x$ :

$$\frac{dy}{dx} \;=\; \frac{e^{3x}\,\sin(2x)}{\sqrt{1+x^2}}\left[\,3 \;+\; \frac{2\cos(2x)}{\sin(2x)} \;-\; \frac{x}{1+x^2}\,\right]$$

## (11). Parametric Differentiation

As we have seen, the graph of a function $y \;=\; f(x)$ is, in general, a curve in 2D space, and we have looked at the geometrical relationship between a function (curve) and its derivative (gradient of tangent to curve).

An implicit relationship between $x$ and $y$ can also be graphed on an *Oxy* axes system to give a curve. An example of this is

$$x^2 \;+\; y^2 \;=\; 9\,;$$

this being the equation of a circle, radius $3$ , centred on the origin. Implicit differentiation would give us the derivative of $y$ with respect to $x$ , and the same geometrical relationship holds.

Another way of describing a curve in 2D space is to use **parametric equations** to define the $x$ and $y$ coordinates of points on the curve. These parametric equations take the form

$$x = f(t) \qquad\qquad y = g(t) \ ,$$

where the $t$ is called a parameter and its value runs over a specified range. Often $t$ is time in which case it may run from 0 to some upper time value. In other cases $t$ may be an angle (usually specified in radians) and has, itself, a geometrical interpretation. For example

$$x = 2\cos t \qquad\qquad y = 2\sin t \quad , \qquad (0 \le t \le 2\pi)$$

are the parametric equations of a circle of radius 2, centred on the origin. We can see this in two ways. Squaring and adding gives

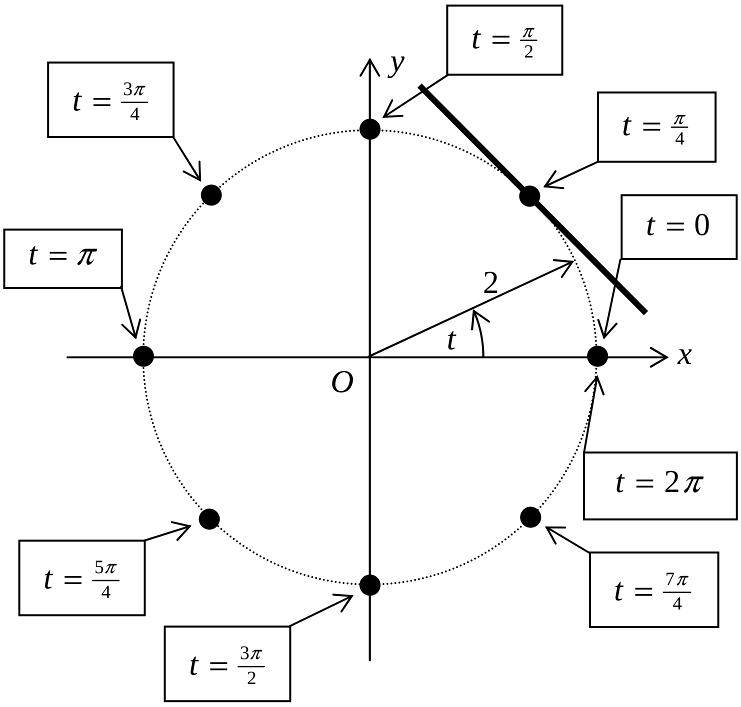
$$\begin{aligned} x^2 + y^2 &= (2\sin t)^2 + (2\cos t)^2 \\ &= 4\sin^2 t + 4\cos^2 t \\ &= 4[\sin^2 t + \cos^2 t] \\ &= 4 \times 1 \\ &= 4 \end{aligned}$$

that is, $x^2 + y^2 = 4$, now recognisable as the equation of a circle.

The other way is to construct a table of values to calculate coordinates of points on the curve, plot the points and join the dots:

| $t$ | $x = 2\cos t$ | $y = 2\sin t$ |
|---|---|---|
| 0 | 2 | 0 |
| $\pi/4$ | $\sqrt{2}$ | $\sqrt{2}$ |
| $\pi/2$ | 0 | 2 |
| $3\pi/4$ | $-\sqrt{2}$ | $\sqrt{2}$ |
| $\pi$ | –2 | 0 |
| $5\pi/4$ | $-\sqrt{2}$ | $-\sqrt{2}$ |
| $3\pi/2$ | 0 | –2 |
| $7\pi/4$ | $\sqrt{2}$ | $-\sqrt{2}$ |
| $2\pi$ | 2 | 0 |

In this context, $t$ is an angle measured (conventionally) anticlockwise from the positive $x$-axis. If these coordinates are now plotted we get the following:




Even though a curve may be define parametrically, we do not need to convert to an implicit or explicit form to determine the derivative of $y$ with respect to $x$. We can use the following formula which is derived from the Chain Rule:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} .$$

**Examples**

**(19).** For the parametric equations above:

$$x = 2\cos t \qquad\qquad y = 2\sin t$$

$$\frac{dx}{dt} = -2\sin t \qquad\qquad \frac{dy}{dt} = 2\cos t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{-2\sin t} = \underline{\underline{-\frac{\cos t}{\sin t}}} .$$

As an example, if we take the parameter value $t = \frac{\pi}{4}$, this gives

$$\frac{dy}{dx} = -1$$

which confirms the orientation of the tangent line at $t = \frac{\pi}{4}$ included in the diagram above.

**(20).** For the parametric equations:

$$x = 2t + 3 \qquad y = t^2$$

$$\frac{dx}{dt} = 2 \qquad \frac{dy}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{2} = \underline{\underline{t}}$$

**(12). Higher Order Derivatives (Notation)**

Before we proceed with some applications of differentiation, let's just look at the notation for the derivative and how it is extended to cover repeated differentiation. We can differentiate a derivative to give a "second derivative" and again to give a "third derivative" and again and again as often as required. We use the following notation:

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d^2y}{dx^2}$$

$$\frac{d}{dx}\left[\frac{d^2y}{dx^2}\right] = \frac{d^3y}{dx^3} \quad ,$$

and so on. If using the $f$ notation:

$$\frac{dy}{dx} = f'(x)$$

$$\frac{d^2y}{dx^2} = f''(x)$$

$$\frac{d^3y}{dx^3} = f'''(x) \quad \text{or} \quad f^{(3)}(x) \quad .$$

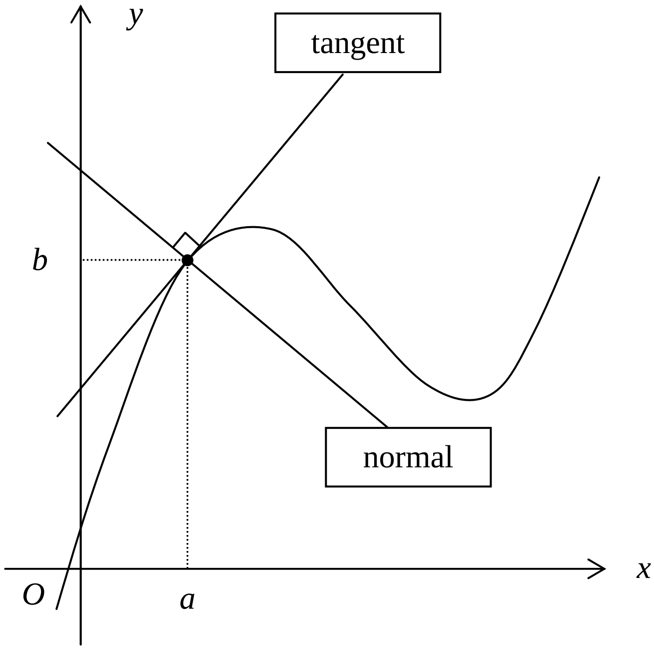
This idea of repeated differentiation crops up in various areas of differential calculus. We have already seen one instance in Example (8) where we looked at displacement, velocity and acceleration:

$$v = \frac{ds}{dt}$$

$$a = \frac{dv}{dt} = \frac{d}{dt}\frac{ds}{dt} = \frac{d^2s}{dt^2} \quad .$$

**(13). Equations of Tangents and Normals**

Whether a curve is defined by an explicit, implicit or parametric form, the tangent to the curve at a given point has a gradient equal to the derivative's value at that point. Related to the tangent is a line known as the **normal** to the curve. This is a line that is perpendicular to the tangent line (see diagram below):




If we denote the gradient of the tangent line by $m_t$ and the gradient of the normal line by $m_n$, we know from coordinate geometry that

$$m_t \times m_n = -1 .$$

For a point on the curve with coordinates $(a , b)$, we have:

equation of tangent line: $y - b = m_t (x - a)$

equation of normal line: $y - b = -\frac{1}{m_t}(x - a)$ ,

where $m_t$ is the derivative evaluated at the point $(a , b)$.

**Examples**

**(21).** For the curve defined by $y = x^2 + 3x + 2$, determine the equations of the tangent and normal lines at the point $(1\,,\,6)$.

First determine the function's derivative:

$$\frac{dy}{dx} = 2x + 3 \ .$$

Next, determine the tangent's gradient $m_t$ by evaluating this derivative at the given point, i.e. $x = 1$:

$$m_t = 2\times 1 + 3 = 5 \ .$$

The gradient of the normal line is therefore

$$m_n = -\tfrac{1}{5} \ .$$

We can now write down the required equations:

tangent:

$$y - 6 = 5(x - 1)$$
$$y - 6 = 5x - 5$$
$$y = 5x + 1$$

normal:

$$y - 6 = -\tfrac{1}{5}(x - 1)$$
$$y - 6 = -\tfrac{1}{5}x + \tfrac{1}{5}$$
$$y = -\tfrac{1}{5}x + \tfrac{31}{5}$$

**(22).** Determine the tangents to the curve $y = x^3 - 7x$ that are parallel to the line $y = 5x - 1$.

At the points of contact of the tangents the curve's gradient must equal that of the given line, i.e. it must equal 5. So we must determine the derivative and set it equal to 5:

$$\frac{dy}{dx} = 3x^2 - 7 = 5 \ .$$

Solve for $x$:

$$3x^2 = 12$$

$$x^2 = 4$$

$$x = \pm 2 \ .$$

Crucially we have two values of $x$ indicating two separate tangents.

**Tangent 1:**

At $x = +2$, $y = 2^3 - 7\times 2 = -6$ .

The point of contact of this tangent is therefore $(2 , -6)$ .

This means that the equation of this first tangent is

$$y - (-6) = 5(x - 2)$$

$$y + 6 = 5x - 10$$

$$\underline{\underline{y = 5x - 16}}$$

**Tangent 2**

At $x = -2$, $y = (-2)^3 - 7\times(-2) = +6$ .

The second point of contact is therefore $(-2 , +6)$ .

The second tangent equation is therefore

$$y - 6 = 5(x - (-2))$$

$$y - 6 = 5x + 10$$

$$\underline{\underline{y = 5x + 16}}$$

**(23).** For the curve defined (implicitly) by $5x^2 - 2xy + 5y^2 = 12$, determine the equations of the tangent and normal lines at the point $(-1, 1)$.

Use implicit differentiation to determine the derivative:

$$\frac{d}{dx}(5x^2 - 2xy + 5y^2) = \frac{d}{dx}(12)$$

$$10x - \left(2 \,.\, y + 2x \,.\, 1\frac{dy}{dx}\right) + 10y\frac{dy}{dx} = 0$$

$$10x - 2y - 2x\frac{dy}{dx} + 10y\frac{dy}{dx} = 0$$

$$(10y - 2x)\frac{dy}{dx} = 2y - 10x$$

$$\frac{dy}{dx} = \frac{2(y - 5x)}{2(5y - x)}$$

$$\frac{dy}{dx} = \frac{y - 5x}{5y - x} \quad .$$

Evaluate the derivative at the point $(-1, 1)$:

$$\frac{dy}{dx} = \frac{1 - 5\times(-1)}{5\times 1 - (-1)} = \frac{6}{6} = 1 \quad .$$

So $m_t = 1$ and $m_n = -1$. The required equations are therefore:

tangent: $y - 1 = 1(x + 1)$

$y - 1 = x + 1$

$y = x + 2$

normal: $y - 1 = -1(x + 1)$

$y - 1 = -x - 1$

$y = -x$

## (14). Curve Sketching and "Max / Min"

As you know, if we have a function $y = f(x)$, then we can plot its graph on an *Oxy* axes system. One way is to construct a table of *x*-values, compute the corresponding *y*-values, plot the pairs of values as points and join them up with a nice, smooth curve. This can be quite laborious and important points of the function can be missed.

An alternative method is to determine just a few **critical** points, enough to allow us to sketch the graph and display any of its important characteristics. The critical points we shall consider are the **intercepts with the axes** and so-called **stationary points**.

### (a) Intercepts with the Axes

Here we find where the graph cuts the horizontal and vertical axes:

**(i).** Determine the *y*-intercept by setting $x = 0$ in $y = f(x)$ and evaluating for *y*. There will only be one value for functions of the form $y = f(x)$.

**(ii).** Determine the *x*-intercepts by setting $y = 0$ in $y = f(x)$ and solving for *x*. There may be none, one or many.

**Example**

**(24).** Determine the axes intercepts of $y = x^2 - x - 2$.

Set $x = 0$: $\quad y = -2$

Set $y = 0$: $\quad x^2 - x - 2 = 0$

Using the quadratic formula

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4.(1).(-2)}}{2.(1)}$$

$$= 2 \quad \text{or} \quad -1 \; .$$

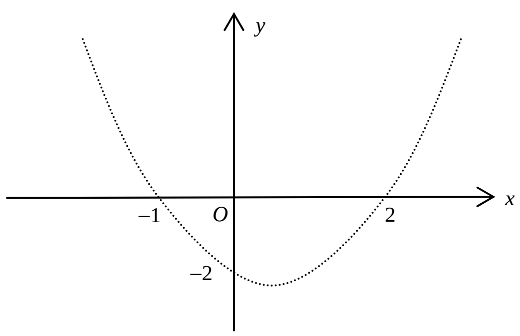



**Figure 4**

## (b). Stationary Points

In the previous example, clearly a critical point is the apex or turning point of the graph. Turning points, points where the graph dips to a minimum or rises to a maximum, are where the graph is neither increasing nor decreasing; we say that the function is **stationary**.

At a stationary point we must have

$$\frac{dy}{dx} = 0 \quad .$$

This gives us a way of precisely locating maxima and minima; set

$$\frac{dy}{dx} = 0$$

and solve for $x$.

To determine the type of stationary point, we can examine the derivative either side of the stationary point to see whether the function is increasing or decreasing (recall p5).

For a stationary point at some value $x = x_0$, there are **four** possible configurations:

### (i). Minimum Turning Point

| | $x < x_0$ | $x = x_0$ | $x > x_0$ |
|---|---|---|---|
| $\frac{dy}{dx}$ | –ve | 0 | +ve |
| Direction of graph | ↘ | → | ↗ |

### (ii). Maximum Turning Point

| | $x < x_0$ | $x = x_0$ | $x > x_0$ |
|---|---|---|---|
| $\frac{dy}{dx}$ | +ve | 0 | –ve |
| Direction of graph | ↗ | → | ↘ |

### (iii). Horizontal Point of Inflection (1)

| | $x < x_0$ | $x = x_0$ | $x > x_0$ |
|---|---|---|---|
| $\frac{dy}{dx}$ | –ve | 0 | –ve |
| Direction of graph | ↘ | → | ↘ |

### (iv). Horizontal Point of Inflection (2)

| | $x < x_0$ | $x = x_0$ | $x > x_0$ |
|---|---|---|---|
| $\frac{dy}{dx}$ | +ve | 0 | +ve |
| Direction of graph | ↗ | → | ↗ |

**Example**

**(25).** Sketch the graph of $y = 3x^5 - 5x^3$.

***y*-intercept** : Set $x = 0$ to give $y = 0$ .

***x*-intercepts** : Set $y = 0$ and solve for *x*.

$$3x^5 - 5x^3 = 0$$

$$x^3(3x^2 - 5) = 0$$

and so $x = 0$ or
$$\begin{aligned} 3x^2 - 5 &= 0 \\ 3x^2 &= 5 \\ x^2 &= 1.6667 \\ x &= \pm 1.29 \end{aligned}$$

Graph cuts horizontal axis at $x = -1.29 , 0 , +1.29$ .

**Stationary points** : $\dfrac{dy}{dx} = 15x^4 - 15x^2$

$$= 15x^2(x^2 - 1)$$

$$= 15x^2(x-1)(x+1)$$

Derivative equals zero when $x = -1\,,\,0\,,\,+1$ .

Now determine the types of stationary points.

**Nature table**:

| | $-2$ | $-1$ | $-0.5$ | $0$ | $+0.5$ | $+1$ | $+2$ |
|---|---|---|---|---|---|---|---|
| $\dfrac{dy}{dx}$ | +ve | 0 | –ve | 0 | –ve | 0 | +ve |
| Direction of graph | ↗ | → | ↘ | → | ↘ | → | ↗ |

Use the original function to compute the *y*-coordinates of the stationary points:

when $x = -1$ , $y = +2$ $\rightarrow$ $(-1\,,+2)$ is a maximum turning point;

when $x = 0$ , $y = 0$ $\rightarrow$ $(0\,,\,0)$ is a horizontal point of inflection;

when $x = +1$ , $y = -2$ $\rightarrow$ $(+1\,,-2)$ is a minimum turning point.

**Graph** :

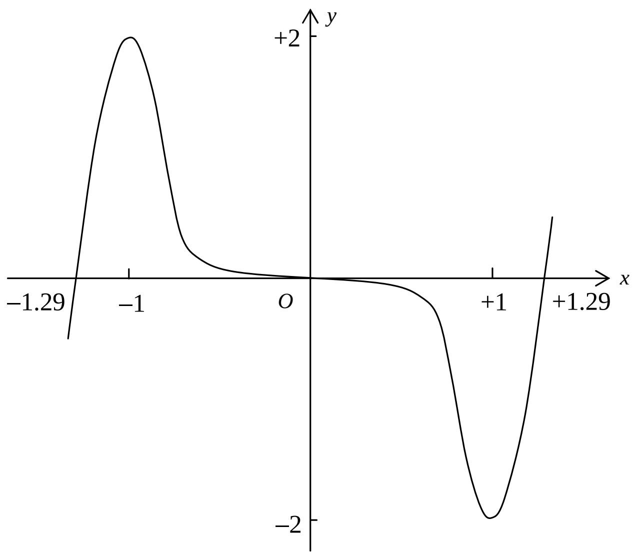
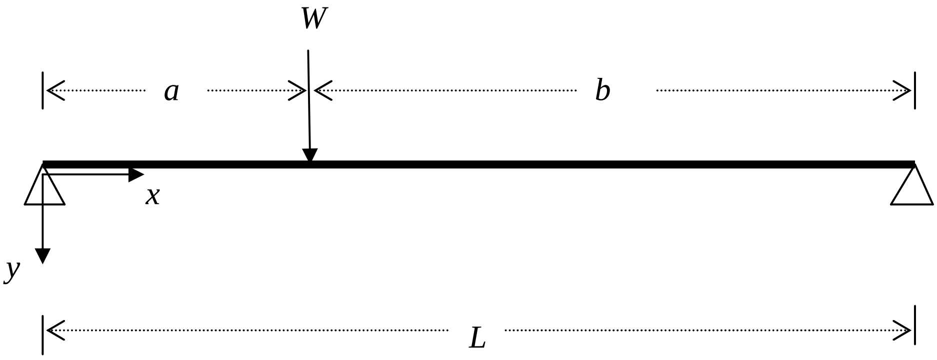



**Figure 5**

When given a function, the aim may not be to sketch its curve, but simply to determine its maximum or minimum values.

**Further Examples**

**(26).** A simply supported beam of length $L$ has a concentrated load $W$, a distance of $a$ metres from the left hand end.




The deflection $y$ at a distance $x$ along the beam is given by

$$y = \frac{Wb}{6EIL}\left\{(L^2 - b^2)x - x^3 + \frac{L}{b}(x - a)^3\right\}$$

where $E$ = Young's modulus and $I$ = moment of inertia of beam.

If $a$ = 2 m, $b$ = 3 m, $L$ = 5 m, determine where the maximum deflection occurs.

Let $\frac{Wb}{6EIL} = k$ so $y = k\left\{(L^2 - b^2)x - x^3 + \frac{L}{b}(x - a)^3\right\}$

Substitute in the values of $a$, $b$ and $L$ to give $y = k\left\{16x - x^3 + \frac{5}{3}(x - 2)^3\right\}$

Hence $\frac{dy}{dx} = k\left\{16 - 3x^2 + 5(x - 2)^2\right\}$

$$= k\left\{16 - 3x^2 + 5(x^2 - 4x + 4)\right\}$$

$$= k\left\{16 - 3x^2 + 5x^2 - 20x + 20\right\}$$

$$= k\left\{2x^2 - 20x + 36\right\}$$

$$= 2k\left\{x^2 - 10x + 18\right\}$$

Stationary points will occur when $x^2 - 10x + 18 = 0$.

Use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with $a = 1$ , $b = -10$ , $c = 18$.

This gives $x = \dfrac{10 \pm \sqrt{100 - 4(1)(18)}}{2} = \dfrac{10 \pm \sqrt{28}}{2}$ .

Hence $x = 7.646$ (which we can rule out as length of beam is only 5 metres) and $x = 2.354$.

The nature table for this stationary point is

| | $x = 2$ | $x = 2.354$ | $x = 3$ |
|---|---|---|---|
| $\dfrac{dy}{dx}$ | $4k$ (+ve) | 0 | $-6k$ (–ve) |
| Tangent line | ↗ | → | ↘ |

which confirms that the stationary point is a maximum.

Maximum deflection occurs when $\underline{\underline{x = 2.354}}$. Substitute this value into

$$y = k\left\{16x - x^3 + \frac{5}{3}(x - 2)^3\right\}$$

to give the maximum deflection as

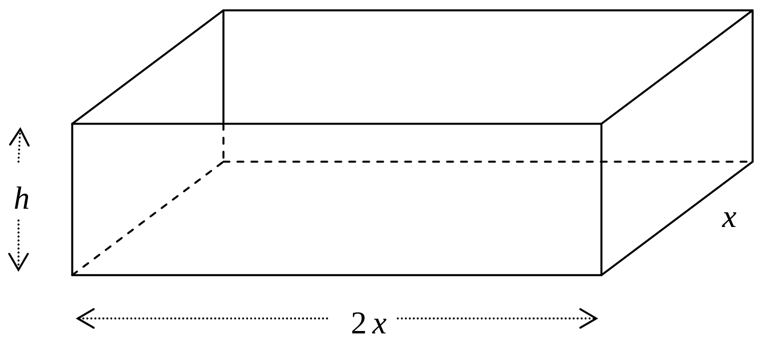
$$\underline{\underline{y = 24.69k}} \; .$$

**(27).** Water tanks are to be manufactured to hold one cubic metre. If the length of each tank is twice its width, what should the dimensions of each tank be to keep the cost of manufacture to a minimum.

We want to use as little material as possible in the manufacture of each tank to keep the cost to a minimum. Hence we must consider the area of material used.

Define the variables as:

Let width of tank be $x$ metres, length of tank be $2x$ metres and the height of tank $h$ metres. A diagram will help in this type of question where the function to be optimised has to be constructed.




Surface area $A$ of tank = bottom + front + back + two sides

$$A = 2x^2 + 2xh + 2xh + xh + xh$$

$$A = 2x^2 + 6xh$$

At present, $A$ is a function of two variables $x$ and $h$. We will have to eliminate one in order to deal with a function of a single variable.

We are told the volume of the tank is $1 \text{ m}^3$ so $2x^2 h = 1 \Rightarrow h = \dfrac{1}{2x^2}$.

Now replace $h$ in terms of $x$ in the above expression for $A$.

$$A = 2x^2 + 6x.\frac{1}{2x^2} = 2x^2 + 3x^{-1}$$

Hence

$$\frac{dA}{dx} = 4x - 3x^{-2} = 4x - \frac{3}{x^2} = \frac{4x^3 - 3}{x^2}$$

The derivative is zero when $4x^3 - 3 = 0 \Rightarrow x^3 = 0.75$

Hence $x = (0.75)^{1/3} = 0.91 \text{ m}$

A nature table will confirm that $A$ is minimised at $x = 0.91$.

The minimum area $A_{\min} = 2(0.91)^2 + \dfrac{3}{0.91} = 4.95 \text{ m}^2$

The height of the tank $h = \dfrac{1}{2x^2} = \dfrac{1}{2(0.91)^2} = 0.6$ metres.

In summary, the dimensions of the tank are: base measurements 0.91 m and 1.82 m and height 0.6 m to give a minimum surface area of 4.95 $\text{m}^2$.

# Tutorial Exercises

## Basic Differentiation

**Q1.** **(a).** Determine the average rate of change of $y = 3x^2 + 1$ for the following changes in $x$:

**(i).** $x: 2 \rightarrow 3$ **(ii).** $x: 2 \rightarrow 2.5$

**(iii).** $x: 2 \rightarrow 2.25$ **(iv).** $x: 2 \rightarrow 2.1$

**(v).** $x: 2 \rightarrow 2.01$ **(vi).** $x: 2 \rightarrow 2.001$ .

**Example:** part (i)

| | | | |
|---|---|---|---|
| $x$ | 2 | 3 | $\Delta x = 3 - 2 = 1$ |
| $y$ | 13 | 28 | $\Delta y = 28 - 13 = 15$ |
| | | | $\frac{\Delta y}{\Delta x} = \frac{15}{1} = 15$ |

**(b).** Using the rule for differentiating $x^n$, determine $f'(x)$ and evaluate it when $x = 2$. Compare your answer with those from part (a).

**Q2.** Differentiate the following functions:

**(i).** $y = 3x + 1$ **(ii).** $y = 2x^2 + 3x - 4$

**(iii).** $y = -6x + 3x^{-1}$ **(iv).** $y = 4x^3 - x^{-2}$

**(v).** $x = 3t^2 - 2t + 1$ **(vi).** $x = (t^4 - t^2)t$ .

## Product Rule

**Q3.** Differentiate the following expressions, simplifying your answers where possible:

**(i).** $y = x^4 \sin x$ **(ii).** $y = x^3 \cos x$

**(iii).** $y = x^2 e^x$ **(iv).** $y = e^x \sin x$

**(v).** $y = x \ln x$ **(vi).** $y = \sin x \tan x$

**(vii).** $y = \cos x \ln x$ **(viii).** $y = e^x \ln x$

**(ix).** $y = (x^2 + 2x + 1)(x - 3)$ **(x).** $y = (x^2 - 1)(x - 1)$ .

**Quotient Rule**

**Q4.** Differentiate the following expressions, simplifying your answers where possible:

**(i).** $y = \dfrac{x^2 - 3x + 2}{x + 1}$ **(ii).** $y = \dfrac{x - 1}{x^2 + 1}$

**(iii).** $y = \dfrac{\sin x}{x}$ **(iv).** $y = \dfrac{\ln x}{x^2}$

**(v).** $y = \dfrac{\sin 2x}{\cos x}$ **(vi).** $y = \dfrac{e^x}{(x + 1)}$ .

**Product Rule and Quotient Rule**

**Q5.** Differentiate the following expressions, simplifying your answers where possible:

**(i).** $y = \dfrac{e^x \cos x}{x}$ **(ii).** $y = \dfrac{(x^2 + 1)\, e^x}{(x + 1)}$ .

**Power Rule / Chain Rule**

**Q6.** Determine the indicated derivative for each of the following functions:

**(i).** $y = 5(x - 7)^4$ : $\dfrac{dy}{dx}$

**(ii).** $y = 2(x^3 - 1)^7$ : $\dfrac{dy}{dx}$

**(iii).** $s = 4(t^3 - 4t + 2)^5$ : $\dfrac{ds}{dt}$

**(iv).** $y = 5(5t - t^2)^3$ : $\dfrac{dy}{dt}$

**(v).** $u = \dfrac{1}{(3t + 1)^6}$ : $\dfrac{du}{dt}$

**(vi).** $R = \dfrac{1}{5(4x^2 - 7)^8}$ : $\dfrac{dR}{dx}$

**(vii). Over the page . . .**

**(vii).** $y = \cos^2 x \quad \{ = (\cos x)^2 \}$ : $\dfrac{dy}{dx}$

**(viii).** $x = (1 + \sin t)^{100}$ : $\dfrac{dx}{dt}$

**(ix).** $y = \sin(100x)$ : $\dfrac{dy}{dx}$

**(x).** $u = e^{-t}$ : $\dfrac{du}{dt}$

**(xi).** $u = e^{t^2}$ : $\dfrac{du}{dt}$

**(xii).** $s = \ln(1 + 2t^3)$ : $\dfrac{ds}{dt}$

**(xii).** $x = \cos(100\pi t + 40)$ : $\dfrac{dx}{dt}$

**(xiv).** $x = \cos^2(100\pi t + 40)$ : $\dfrac{dx}{dt}$

**(xv).** $u = 4\sin^3(60x - 20)$ : $\dfrac{du}{dx}$ .

**Implicit Differentiation**

**Q7.** Using implicit differentiation, determine $\dfrac{dy}{dx}$ from each of the following equations:

**(i).** $y^2 = x^2$

**(ii).** $x^2 - y^2 = 1$

**(iii).** $5x^2 + 7y^2 - 4 = 0$

**(iv).** $2x^2 - 3y^3 = 6$

**(v).** $x\,y = 2$

**(vi).** $2\,x\,y^2 = 5$

**(vii).** $xy + 3x = 2$

**(viii).** $x^2\,y - xy^2 + y = 4$ .

## Logarithmic Differentiation

**Q8.** Use logarithmic differentiation to obtain the derivative $\frac{dy}{dx}$ from the following functions:

**(i).** $y = 3^x$

**(ii).** $y = 2^{x+1}$

**(iii).** $y = \left(x^2 + 1\right)^{x^2}$

**(iv).** $y = \dfrac{x^2(5 + x)^3}{(3 + x)^5}$

**(v).** $y = \dfrac{x^3\sqrt{x - 1}}{(x + 4)^2}$

**(vi).** $y = \dfrac{x^{1/2}(3 - x)^{1/6}}{(2x + 1)^{2/3}}$

**(vii).** $y = xe^{3x}\sin x$

**(viii).** $y = \cos(x)\cos(2x)\cos(3x)$

## Parametric Differentiation

**Q9.** Find the first derivative $\frac{dy}{dx}$ for the following parametric equations :

**(i).** $x = t \ , \ y = \sqrt{t}$

**(ii).** $x = 6t^2 \ , \ y = 4t^3 - 3t$

**(iii).** $x = t + \dfrac{1}{t} \ , \ y = t - \dfrac{1}{t}$

**(iv).** $x = 4\cos t \ , \ y = 2\sin t$

**(v).** $x = \dfrac{4t}{(1 + t^2)^2} \ , \ y = \dfrac{4t^2}{(1 + t^2)^2}$

**Q10.** Obtain the gradient of the curve defined by the following parametric equations at the point indicated :

**(i).** $x = 2t^2 + 3 \ , \ y = t^4$ at the point where $t = -1$

**(ii).** $x = -\sqrt{t + 1} \ , \ y = \sqrt{(3t)}$ at the point where $t = 3$

**(iii).** $x = \tan t \ , \ y = \sin^2 t$ at the point where $t = \dfrac{\pi}{4}$

**Q11**. Given the parametric equations

$$x = 2e^{3t} \ , \quad y = 1 + \sin^2 t$$

show that

$$3x\frac{dy}{dx} = 2\sqrt{(y - 1)(2 - y)}\,.$$

**Repeated Differentiation**

**Q12**. Determine $\frac{d^2y}{dx^2}$ when

**(i).** $y = 2x^3 - 3x^2 + 1$

**(ii).** $y = \sin(2x)$

**(iii).** $y = \cos(3x)$

**(iv).** $y = e^{-4x}$ .

**Equations of Tangents and Normals**

**Q13.** Find the equations of the tangents and normals to the following curves at the points indicated :

**(i).** $y = x^2 - 2x \ , \ x = 0$

**(ii).** $y = (x - 1)(2x - 1) \ , \ y = 0$

**(iii).** $y = (2x - 1)^2 \ , \ x = 0.5$

**(iv).** $x^2 + 4y^2 = 5 \ , \ (1, -1)$

**(v).** $y^2 = 8x \ , \ (2, -4)$

**(vi).** $x^2 + 2y - 3xy = 2 \ , \ (0, 1)$

**(vii).** $x^2 + y^2 - 8x + 6y = 0 \ , \ (7, 1)$

**(viii).** $x = 2\cos t, \quad y = 3\sin t, \quad t = \frac{\pi}{4}$

**(ix).** $x = 4t^2 + t + 2 \ , \ y = t^2 - 2t + 3 \ , \ t = 0$

**(x).** $x = 2\sin t, \quad y = \cos 2t, \quad t = \frac{\pi}{6}$.

**Q14.** Show that $y = x^3 - 3x^2 + 2x + 3$ has gradient 2 at two points, one of which lies on $y = 2x - 1$.

Hence show that $y = 2x - 1$ intersects $y = x^3 - 3x^2 + 2x + 3$ at two points only.

**Q15.** The curve $y = x(1 - x^2)$ intersects the line $x - 3y = 0$ at three points. Show that the tangents to the curve at two of these points are (respectively) parallel and perpendicular to the tangent at the third point.

**Curve Sketching**

**Q16.** Sketch the graphs of the following functions, showing clearly axes intercepts, maxima, minima and horizontal points of inflection:

**(i).** $y = x^3 - x^2 - x$

**(ii).** $y = x^4 - x^3 - 2x^2$

**(iii).** $y = x^3 + 1$

**(iv).** $y = 3x^4 - 4x^3$ .

**Q17.** Determine the locations and types of the stationary points for the function $y = x^2 e^{-x}$.

**Max / Min**

**Q18.** The speed , $v$ , of a car (in m/s) is related to time t seconds by the equation

$$v = 3 + 12t - 3t^2 \quad .$$

Determine the maximum speed of the car in **kilometres per hour**.

**Q19.** The deflection $D$ (in $\mu$m ) of a beam of length 10 metres is given by

$$D = 2x^4 - 50x^3 + 300x^2$$

where $x$ is the distance from one end of the beam. Find the value of $x$ that yields the maximum deflection.

**Hint:** You will need the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

**Q20.** An open water tank on a square base of length *x* m is to be made from sheet metal using 2 square metres of material. Show that the volume *V* of the tank is given by

$$V = \frac{1}{2}x - \frac{1}{4}x^3 .$$

Determine the dimensions of the tank that will give a maximum volume.

**Q21.** Cylindrical water tanks of radius r m are to be constructed to hold 0.8 cubic metres. Show that the area *A* of the material used in manufacturing an open cylindrical tank is given by

$$A = \frac{1.6}{r} + \pi r^2 .$$

Determine the value of *r* which minimises *A*.

**Q22.** An open-topped rectangular tank of volume 36 $m^3$ is to be constructed. The base of the tank has one side twice as long as the other. If the shorter side of the base has length *x* metres :

**(i).** show that the surface area of the tank, *A* $m^2$, is given by

$$A = 2x^2 + \frac{108}{x} .$$

**(ii).** determine the dimensions of the tank which minimises *A*.

## Answers

**A1.** **(a).** **(i).** 15 **(ii).** 13.5 **(iii).** 12.75

**(iv).** 12.3 **(v).** 12.03 **(vi).** 12.003

**(b).** 12

**A2.** **(i).** $3$ **(ii).** $4x + 3$

**(iii).** $-6 - 3x^{-2}$ **(iv).** $12x^2 + 2x^{-3}$

**(v).** $6t - 2$ **(vi).** $5t^4 - 3t^2$

**A3.** **(i).** $4x^3 \sin x + x^4 \cos x$ **(ii).** $3x^2 \cos x - x^3 \sin x$

**(iii).** $(2x + x^2)\, e^x$ **(iv).** $e^x (\sin x + \cos x)$

**(v).** $1 + \ln x$ **(vi).** $\cos x \tan x + \sin x \sec^2 x$

**(vii).** $\dfrac{\cos x}{x} - \sin x \ln x$ **(viii).** $e^x \left[ \ln x + \dfrac{1}{x} \right]$

**(ix).** $3x^2 - 2x - 5$ **(x).** $3x^2 - 2x - 1$

**A4.** **(i).** $\dfrac{x^2 + 2x - 5}{(x+1)^2}$ **(ii).** $\dfrac{1 + 2x - x^2}{(x^2+1)^2}$

**(iii).** $\dfrac{x \cos x - \sin x}{x^2}$ **(iv).** $\dfrac{1 - 2\ln x}{x^3}$

**(v).** $\dfrac{2\cos 2x \cos x + \sin 2x \sin x}{\cos^2 x}$ **(vi).** $\dfrac{x\, e^x}{(x+1)^2}$

**A5.** **(i).** $\dfrac{e^x\, [(x-1)\cos x - x \sin x]}{x^2}$ **(ii).** $\dfrac{x\,(x^2 + 2x + 3)\, e^x}{(x+1)^2}$

**A6.** **(i).** $20\,(x-7)^3$

**(ii).** $42\,x^2\,(x^3-1)^6$

**(iii).** $20\,(3t^2-4)\,(t^3-4t+2)^4$

**(iv).** $15\,(5-2t)\,(5t-t^2)^2$

**(v).** $-\dfrac{18}{(3t+1)^7}$

**(vi).** $-\dfrac{64x}{5\,(4x^2-7)^9}$

**(vii).** $-2\sin x\cos x$

**(viii).** $100\cos t\,(1+\sin t)^{99}$

**(ix).** $100\cos(100x)$

**(x).** $-e^{-t}$

**(xi).** $2\,t\,e^{t^2}$

**(xii).** $\dfrac{6t^2}{1+2t^3}$

**(xiii).** $-100\,\pi\sin(100\pi t+40)$

**(xiv).** $-200\,\pi\cos(100\pi t+40)\sin(100\pi t+40)$

**(xv).** $720\cos(60x-20)\sin^2(60x-20)$.

**A7.** **(i).** $\dfrac{x}{y}$

**(ii).** $\dfrac{x}{y}$

**(iii).** $-\dfrac{5x}{7y}$

**(iv).** $\dfrac{4x}{9y^2}$

**(v).** $-\dfrac{y}{x}$

**(vi).** $-\dfrac{y}{2x}$

**(vii).** $-\dfrac{(y+3)}{x}$

**(vii).** $\dfrac{y\,(y-2x)}{x^2-2xy+1}$.

**A8.** **(i).** $3^x\ln(3)$

**(ii).** $2^{x+1}\ln(2)$

**(iii).** $2x\left(x^2+1\right)^{x^2-1}\left((x^2+1)\ln(x^2+1)+x^2\right)$

**(iv).** $\dfrac{x^2(5+x)^3}{(3+x)^5}\left(\dfrac{2}{x}+\dfrac{3}{5+x}-\dfrac{5}{3+x}\right)\equiv\dfrac{30x(5+x)^2}{(3+x)^6}$

**(v).** $\dfrac{x^3\sqrt{x-1}}{(x+4)^2}\left(\dfrac{3}{x}+\dfrac{1}{2(x-1)}-\dfrac{2}{x+4}\right)\equiv\dfrac{x^2(3x^2+26x-24)}{2(x+4)^3\sqrt{x-1}}$

**(vi).** $\dfrac{x^{1/2}(3-x)^{1/6}}{(2x+1)^{2/3}}\left(\dfrac{1}{2x}-\dfrac{1}{6(3-x)}-\dfrac{4}{3(2x+1)}\right)$

$$\equiv\dfrac{9-10x}{6x^{1/2}(3-x)^{5/6}(2x+1)^{5/3}}$$

**(vii).** $xe^{3x}\sin x\left(\dfrac{1}{x}+3+\dfrac{\cos x}{\sin x}\right)\equiv e^{3x}\left(\sin x+3x\sin x+x\cos x\right)$

**(viii).** $\cos(x)\cos(2x)\cos(3x)\left(-\dfrac{2\sin(2x)}{\cos(2x)}-\dfrac{3\sin(3x)}{\cos(3x)}-\dfrac{\sin x}{\cos x}\right)$

which simplifies to

$-\cos(x)\cos(2x)\cos(3x)\left(2\tan(2x)+3\tan(3x)+\tan(x)\right)$

**A9.** **(i).** $\dfrac{1}{2\sqrt{t}}$ **(ii).** $\dfrac{4t^2-1}{4t}$

**(iii).** $\dfrac{1+\dfrac{1}{t^2}}{1-\dfrac{1}{t^2}}\equiv\dfrac{t^2+1}{t^2-1}$ **(iv).** $-\dfrac{1}{2}\cot t$

**(v).** $\dfrac{2t(1-t^2)}{1-3t^2}$.

**A10.** **(i).** At $t=-1$, $\dfrac{dy}{dx}=1$

**(ii).** At $t=3$, $\dfrac{dy}{dx}=-2$

**(iii).** At $t=\dfrac{\pi}{4}$, $\dfrac{dy}{dx}=\dfrac{1}{2}$

**A12.** **(i).** $12x - 6$

**(ii).** $-4\sin(2x)$

**(iii).** $-9\cos(3x)$

**(iv).** $16e^{-4x}$

**A13.**

| | **TANGENT** | **NORMAL** |
|---|---|---|
| **(i).** | $y + 2x = 0$ | $2y = x$ |
| **(ii).** | $y - x + 1 = 0$<br>$y + x - 0.5 = 0$ | $y + x = 1$<br>$y - x = -0.5$ |
| **(iii).** | $y = 0$ | $x = 0.5$ |
| **(iv).** | $4y = x - 5$ | $y = -4x + 3$ |
| **(v).** | $y + x + 2 = 0$ | $y - x + 6 = 0$ |
| **(vi).** | $y = 1.5x + 1$ | $3y = -2x + 3$ |
| **(vii).** | $4y = -3x + 25$ | $3y = 4x - 25$ |
| **(viii).** | $2y + 3x = 6\sqrt{2}$ | $3\sqrt{2}\,y - 2\sqrt{2}\,x = 5$ |
| **(ix).** | $y + 2x = 7$ | $2y - x = 4$ |
| **(x).** | $y + x = \frac{3}{2}$ | $-y + x = \frac{1}{2}$ |

**A16.** **(i).** Vertical intercept: $y = 0$

Horizontal intercepts: $x = \dfrac{1 \pm \sqrt{5}}{2}$ , $x = 0$

Maximum: $\left(-\frac{1}{3}, \frac{5}{27}\right)$

Minimum: $(1, -1)$

**A16.** **(ii).** Vertical intercept: $y = 0$

Horizontal intercepts: $x = -1 \quad , \quad x = 0 \quad , \quad x = 2$

Maximum: $(0 , 0)$

Minima: $(-0.693 , -0.397) \quad , \quad (1.443 , -2.833)$

**(iii).** Vertical intercept: $y = 1$

Horizontal intercept: $x = -1$

HPI: $(0 , 1)$

**(iv).** Vertical intercept: $y = 0$

Horizontal intercepts: $x = 0 \quad , \quad x = \frac{4}{3}$

HPI: $(0 , 0)$

Minimum: $(1 , -1)$

**A17.** Min at $(0 , 0)$ and Max at $(2 , 4e^{-2})$.

**A18.** Maximum speed is 15 m/s, which is 54 Km/hour.

**A19.** $x$ = 5.785 metres.

**A20.** $0.82 \times 0.82 \times 0.41$ m

**A21.** $A_{\text{min}} = 3.79\,\text{m}^2$ when $r = 0.634\,\text{m}$.

**A22.** **(ii).** $3 \times 6 \times 2$ m