Glasgow Caledonian University

School of Computing, Engineering & Built Environment

Mathematics Summer School

Level 2 Entry – Engineering

Complex Numbers

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Complex Numbers

1) The Familiar Number System

The number system we use today did not arise suddenly as the blinding flash of inspiration of a single person. Concepts of number and notation evolved gradually over several millennia, with evolutionary steps often occurring out of the need to answer questions and solve problems. Before we begin this section in earnest, it is useful to look at how our number system is made up from different sets of numbers.

The Natural Numbers ($\mathbb{N}$)

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$

This set is fine for basic counting and it is said to be "closed" under addition and multiplication. That is, add or multiply two natural numbers and you still get a natural number. It doesn't cope that well with subtraction or division.

The Whole Numbers ($\mathbb{W}$)

$$\mathbb{W} = \{0, 1, 2, 3, 4, 5, \ldots\}$$

The zero improves matters slightly; we can now subtract a natural number from itself!

The Integers ($\mathbb{Z}$)

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Subtraction is now within the scope of integers, but division is limited.

The Rational Numbers ($\mathbb{Q}$)

The set of numbers that can be expressed as ratios of two integers. Rational numbers are "closed" under addition, subtraction, multiplication and division, however it does not include solutions to equations like $x^2 - 2 = 0$ or answer a whole host of other mathematical questions, e.g. "What is the ratio of a circle's circumference to its diameter?" Numbers like $\sqrt{2}$ and π are called **Irrational Numbers**.

When all the irrational numbers are included along with the rational numbers, we have the set of so-called **Real Numbers ($\mathbb{R}$)**. This seems to have completed the evolutionary process and provided us with a set of numbers that can deal with any numerical problem. This, as you will see, it not the case. We are now going to extend the number system even further by delving into the realms of **Imaginary Numbers** and **Complex Numbers**.

2) Imaginary Numbers

Consider the equation

$$z^2 + 9 = 0 \ .$$

Attempting to solve this equation we obtain

$$z^2 = -9$$

$$z = \pm\sqrt{-9} \ .$$

We appear to have a problem with the square root of a negative number. Do we stop here? Do we give up and say that there is no solution to the problem? Absolutely not! We can write

$$\begin{aligned} z &= \pm\sqrt{(-1)\times 9} \\ &= \pm\sqrt{-1}\times\sqrt{9} \\ &= \pm\sqrt{-1}\times 3 \\ &= \pm 3\sqrt{-1} \ . \end{aligned}$$

We are, however, stuck with evaluating $\sqrt{-1}$ within the set of real numbers, but we can extend our number system to include it. Mathematicians refer to $\sqrt{-1}$ by the lower-case letter i; because engineers use i for current, they usually refer to it by j instead. This means that the solution to our equation

$$z^2 + 9 = 0$$

can be written as

$$z = \pm 3j \ .$$

We can reduce the square root of any negative number in a similar fashion:

$$\sqrt{-n} = j\sqrt{n} \ .$$

Multiples of j are called **Imaginary Numbers**.

3) Complex Numbers (ℂ)

Now consider the equation

$$z^2 + 4z + 13 = 0 \ .$$

This is a quadratic equation. Applying the well-known quadratic formula we obtain

$$z = \frac{-4 \pm \sqrt{4^2 - 4\times1\times13}}{2} = \frac{-4 \pm \sqrt{-36}}{2} \ .$$

Before, we may have stopped at this point and claimed "no solution". However, with the concept of imaginary numbers we can take this further:

$$z = \frac{-4 \pm 6j}{2}$$

$$= -2 \pm 3j$$

$$= -2 + 3j \quad \text{or} \quad -2 - 3j \ .$$

We have two solutions of the quadratic equation, each of which appears to be a combination of real and imaginary numbers. We call such a combination

$$a + bj \ ,$$

where a and b are real numbers, a **Complex Number**.

Notes

- In $a + bj$, a is called the **real part** and b the **imaginary part** of the complex number.

  e.g. $5 - 2j$: real part is 5; imaginary part is -2.

- Two complex numbers are equal if, and only if, their real parts are equal **and** their imaginary parts are equal.

- The real numbers are a subset of the complex numbers: e.g. $4 = 4 + 0j$. So a real number may be regarded as a complex number with a zero imaginary part.

- Similarly, the imaginary numbers are also a subset of the complex numbers: e.g. $3j = 0 + 3j$. So an imaginary number may be regarded as a complex number with a zero real part.

- Although the concept of complex numbers may seem a totally abstract one, complex numbers have many real-life applications in applied mathematics and engineering.

4) The Arithmetic of Complex Numbers

All the usual arithmetic operations associated with real numbers can be performed on complex numbers. Whatever the operation or combination of operations, the answer can always be written in the form $a + bj$.

When dealing with complex arithmetic, it is good practice to write complex numbers in brackets. The brackets can then be removed using usual algebraic techniques.

a) Addition and Subtraction

All we do here is combine the real parts and then combine the imaginary parts.

Examples

(1). Given two complex numbers $z_1 = 6 + 3j$ and $z_2 = 8 - 5j$, determine

(i). $z_1 + z_2$;

(ii). $z_1 - z_2$.

(i).
$$\begin{aligned} z_1 + z_2 &= (6 + 3j) + (8 - 5j) \\ &= 6 + 3j + 8 - 5j \\ &= 6 + 8 + 3j - 5j \\ &= \underline{\underline{14 - 2j}} \end{aligned}$$

(ii).
$$\begin{aligned} z_1 - z_2 &= (6 + 3j) - (8 - 5j) \\ &= 6 + 3j - 8 + 5j \\ &= 6 - 8 + 3j + 5j \\ &= \underline{\underline{-2 + 8j}} \end{aligned}$$

b) Multiplication

Note: $j = \sqrt{-1} \quad \rightarrow \quad j^2 = -1$ **.**

Multiplication of two complex numbers is just the same as multiplying out two sets of brackets in ordinary algebra. Just remember that when j^2 appears, we can replace it by -1 .

Example

(2). For $z_1 = 3 + 7j$ and $z_2 = 4 - 5j$, form the product $z_1 z_2$.

$$\begin{aligned} z_1 z_2 &= (3+7j)(4-5j) \\ &= 12 - 15j + 28j - 35j^2 \\ &= 12 + 13j - 35(-1) \\ &= 12 + 13j + 35 \\ &= \underline{\underline{47 + 13j}} \end{aligned}$$

c) Division

The division of one complex number by another is a little more complicated. First note the following.

For any complex number, we form its **complex conjugate** partner by changing the sign of the imaginary part. For example:

complex number: $2 + 3j$
complex conjugate: $2 - 3j$;

complex number: $-4 - 2j$
complex conjugate: $-4 + 2j$.

When a complex number is multiplied by its conjugate, the result is always a positive, real number:

$$(2+3j)(2-3j) = 4 - 6j + 6j - 9j^2 = 4 + 9 = 13$$

$$(-4-2j)(-4+2j) = 16 - 8j + 8j - 4j^2 = 16 + 4 = 20 \ .$$

We use this property to help us divide complex numbers.

Example

(3). For $z_1 = 4 - 5j$ and $z_2 = 2 + 3j$, form the quotient (ratio) $\dfrac{z_1}{z_2}$.

$$\frac{z_1}{z_2} = \frac{(4 - 5j)}{(2 + 3j)}$$

Complex fractions are no different from real number fractions in that you can multiply top (numerator) and bottom (denominator) by the same number and its "net value" remains unaltered. Here we choose to multiply top and bottom by the conjugate of the denominator:

$$\frac{z_1}{z_2} = \frac{(4 - 5j)}{(2 + 3j)}\,\frac{(2 - 3j)}{(2 - 3j)} \quad .$$

Next, we multiply out the numerators, and then the denominators:

$$\frac{z_1}{z_2} = \frac{8 - 12j - 10j + 15j^2}{4 - 6j + 6j - 9j^2}$$

$$= \frac{8 - 22j - 15}{4 + 9}$$

$$= \frac{-7 - 22j}{13}$$

$$= \underline{\underline{-\frac{7}{13} - \frac{22}{13}j}} \quad .$$

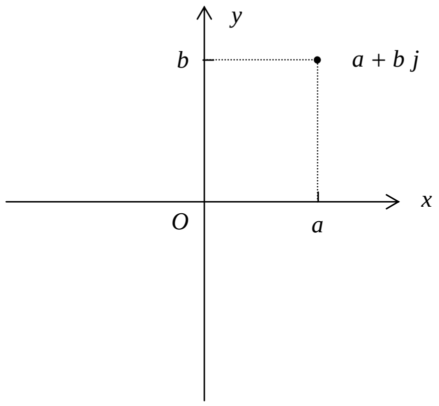
Multiplying top and bottom by the conjugate of the denominator will always give a single real number in the denominator position, and so the division can be completed.

d) Powers and Roots of Complex Numbers

To complete the basic arithmetic of complex numbers we shall look at determining powers and roots. However, we shall defer this until Section 6, after we have looked at an alternative representation for complex numbers.

5) The Rectangular Form and Polar Form of a Complex Number

As we have seen, a complex number is specified by two "ordinary" numbers, the real part and the imaginary part. By regarding these two numbers as coordinates on an *Oxy* axes system, we can represent a complex number graphically by a point:

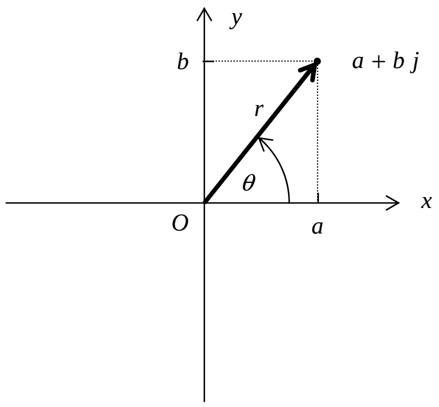


In this context, the x-axis is called the **real axis**, the y-axis is the **imaginary axis** and the whole axes system is an **Argand diagram**. Given this link to coordinates, we shall now refer to

$$a \; + \; b\,j$$

as the **Cartesian** or **rectangular form** of a complex number.

If we now indicate the position of a point depicting a complex number by an arrow radiating from the origin, that is,



we can use the arrow length (r) and orientation (θ) as an alternative way of specifying a complex number. This gives us the so-called **polar form** of a complex number which is written as either

$$z \;=\; r\,(\cos\theta^\circ \;+\; j\,\sin\theta^\circ)$$

or the abbreviated version

$$z \;=\; r\angle\,\theta^\circ \;\; ;$$

in this, r is called the **magnitude** of the complex number, and θ°, its **angle** or **argument**.

6) The Relationship Between Polar and Cartesian (Rectangular) Forms

Rectangular Form: $z = a + b\,j$

Polar Form: $z = r \angle \theta°$

A combination of basic trigonometry and Pythagoras' Theorem gives the following conversion formulae:

Polar $\rightarrow$ Rectangular: $a = r\cos\theta°$ $\quad b = r\sin\theta°$

Rectangular $\rightarrow$ Polar: $r = |\,z\,| = \sqrt{a^2 + b^2}$ $\quad \theta = \tan^{-1}\left(\dfrac{b}{a}\right)$.

The conversion "Polar $\rightarrow$ Rectangular" is quite straightforward, but care must be taken when applying "Rectangular $\rightarrow$ Polar", since the quadrant in which θ lies must be determined before evaluating the inverse tangent.

Examples

(4). (i). A complex number has magnitude 2 and angle $210°$. Express the complex number in its Cartesian or rectangular form.

$$z = r \angle \theta° = 2 \angle 210°$$

$$\begin{aligned} a &= r\cos\theta° \\ &= 2\cos(210°) \\ &= -1.732 \end{aligned} \qquad \begin{aligned} b &= r\sin\theta° \\ &= 2\sin(210°) \\ &= -1 \end{aligned}$$

$$\underline{\underline{z = -1.732 - 1\,j}}$$

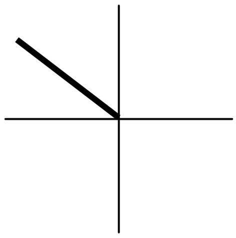
(ii). Express the complex number $z = -4 + 2\,j$ in polar form:

$$z = a + b\,j = -4 + 2\,j$$

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ &= \sqrt{(-4)^2 + 2^2} \\ &= \sqrt{20} \\ &= 4.472 \end{aligned}$$

Determine the / . . .

Determine the quadrant for the angle ($a = -4$, $b = 2$) :



$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$= \tan^{-1}\left(\frac{2}{-4}\right) \qquad [\text{ 2nd quadrant angle}]$$

$$= 153.43°$$

$$\underline{\underline{z = 4.472 \angle 153.43°}}$$

Most calculators have these conversion formulae pre-programmed. Please refer to your own calculator's instruction booklet for information on how to implement these conversion processes or, if that fails, ask in the tutorial classes.

7) The Arithmetic of Complex Numbers in Polar Form

Addition and subtraction is only really feasible in Cartesian (rectangular) form. However, other aspects of complex arithmetic are simplified in polar form.

a) Multiplication and Division

If we have two complex numbers in polar form,

$$z_1 = r_1 (\cos\theta_1 + j\sin\theta_1) = r_1 \angle \theta_1$$

$$z_2 = r_2 (\cos\theta_2 + j\sin\theta_2) = r_2 \angle \theta_2$$

then, by some application of trig identities, it can be shown that their product and quotient are given by

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2)] = r_1 r_2 \angle (\theta_1 + \theta_2)$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + j\sin(\theta_1 - \theta_2)] = \frac{r_1}{r_2} \angle (\theta_1 - \theta_2) \quad .$$

Examples

(5). Given the two complex numbers in polar form,

$$z_1 = 6 \angle 40° \quad \text{and} \quad z_2 = 2 \angle 30° \ ,$$

determine the product $z_1 z_2$ and quotient $\dfrac{z_1}{z_2}$ also in polar form.

$$z_1 z_2 = r_1 r_2 \angle (\theta_1 + \theta_2)$$

$$= 6 \times 2 \angle (40° + 30°)$$

$$= \underline{\underline{12 \angle 70°}}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle (\theta_1 - \theta_2)$$

$$= \frac{6}{2} \angle (40° - 30°)$$

$$= \underline{\underline{3 \angle 10°}}$$

(6). Similarly for $z_1 = 10 \angle 80°$ and $z_2 = 4 \angle (-30°)$,

$$z_1 z_2 = r_1 r_2 \angle (\theta_1 + \theta_2)$$

$$= 10 \times 4 \angle (80° + (-30°))$$

$$= \underline{\underline{40 \angle 50°}}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle (\theta_1 - \theta_2)$$

$$= \frac{10}{4} \angle (80° - (-30°))$$

$$= \underline{\underline{2.5 \angle 110°}}$$

b) Powers of Complex Numbers

For

$$z = r(\cos\theta + j\sin\theta) = r \angle \theta,$$

we can compute a power of z using the formula

$$z^n = r^n(\cos n\theta + j\sin n\theta) = r^n \angle n\theta \ .$$

This not obvious but perhaps can be seen if we look at a couple of simple cases and link back to the multiplication rule of the previous subsection:

$$z = r \angle \theta$$

$$z^2 = z.z = r.r \angle (\theta + \theta) = r^2 \angle 2\theta$$

$$z^3 = z.z^2 = r.r^2 \angle (\theta + 2\theta) = r^3 \angle 3\theta \ .$$

Examples

(7). **(i).** $z = 2 \angle 40°$ $\rightarrow$ $z^3 = 2^3 \angle 3 \times 40° = 8 \angle 120°$

(ii). $z = 5 \angle (-20°)$ $\rightarrow$ $z^2 = 5^2 \angle 2 \times (-20°) = 25 \angle (-40°)$

c) Roots of Complex Numbers

When working solely with ordinary (real) numbers, if we take a square root we obtain either two values (if the number is positive) or no values (if the number is negative). For example,

$$\sqrt{9} = +3 \quad \text{or} \quad \sqrt{9} = -3 \ ;$$

$\sqrt{-9}$ not possible in the real number system.

Extending our number system to include complex numbers will allow us to determine two square roots for all numbers, positive, negative or, indeed, complex numbers themselves.

In the real number system numbers have only one cube root, e.g.

$$\sqrt[3]{8} = 2 \quad , \quad \sqrt[3]{-27} = -3 \ ;$$

in the complex number system a number has three cube roots.

And the pattern continues. In general, determining the ***n*th roots** of a number will yield ***n* values** when working in the complex number system.

To determine the roots of a number z , we first ensure it is expressed as a polar complex number:

$$z = r \angle \theta \ .$$

Just as in the real number system, roots can be expressed as fractional powers. That is

$$\sqrt{z} \equiv z^{1/2}$$

$$\sqrt[3]{z} \equiv z^{1/3}$$

$$\sqrt[n]{z} \equiv z^{1/n} \ ,$$

where $\equiv$ means "identical to". This being the case, we can use the result from the "Powers" subsection above to determine roots:

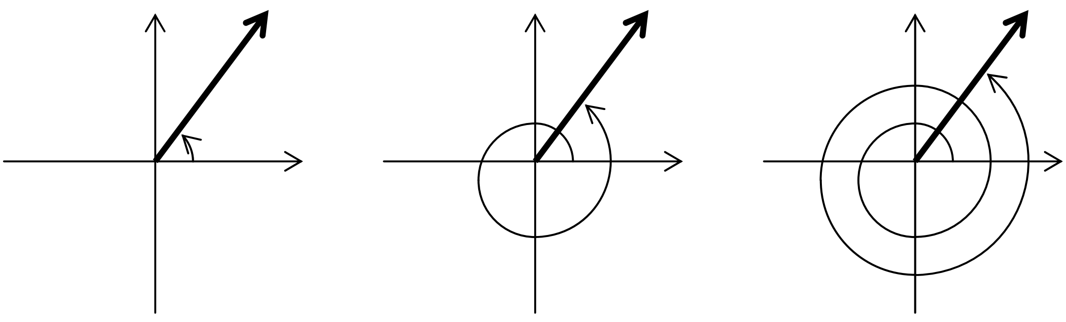
$$z = r \angle \theta$$

$$z^{1/n} = r^{1/n} \angle \left(\frac{\theta}{n} \right) .$$

This will give us one *n*th root. What about the other $n-1$? Note that on an Argand diagram

$$z = r \angle \theta$$
$$z = r \angle (\theta + 360°)$$
$$z = r \angle (\theta + 2\times 360°)$$
etc.

all occupy the same position:



To find all the roots of a complex number, we must consider these added rotations.

Square Roots

Number: $z \;=\; r \angle \theta$

Root 1: $z^{1/2} \;=\; r^{1/2} \angle \left(\dfrac{\theta}{2} \right)$

Root 2: $z^{1/2} \;=\; r^{1/2} \angle \left(\dfrac{\theta + 360°}{2} \right) \;=\; r^{1/2} \angle \left(\dfrac{\theta}{2} + 180° \right)$

Note that the magnitudes of the two roots are the same and the angle increment is 180°.

Example

(8). Number: $z \;=\; 3 \;+\; 4\,j \;=\; 5 \angle 53.13°$ [Two square roots]

Root 1: $z^{1/2} \;=\; 5^{1/2} \angle \left(\dfrac{53.13°}{2} \right) \;=\; 2.24 \angle 26.56°$

Root 2: $z^{1/2} \;=\; 5^{1/2} \angle \left(\dfrac{53.13° + 360°}{2} \right) \;=\; 2.24 \angle \left(26.56° + 180° \right)$

$\;=\; 2.24 \angle 206.56°$

Cube Roots

Number: $z \;=\; r \angle \theta$

Root 1: $z^{1/3} \;=\; r^{1/3} \angle \left(\dfrac{\theta}{3} \right)$

Root 2: $z^{1/3} \;=\; r^{1/3} \angle \left(\dfrac{\theta + 360°}{3} \right) \;=\; r^{1/3} \angle \left(\dfrac{\theta}{3} + 120° \right)$

Root 3: $z^{1/3} \;=\; r^{1/3} \angle \left(\dfrac{\theta + 2 \times 360°}{3} \right) \;=\; r^{1/3} \angle \left(\dfrac{\theta}{3} + 2 \times 120° \right) \;=\; r^{1/3} \angle \left(\dfrac{\theta}{3} + 240° \right)$

Note that the magnitudes of the three roots are the same and the angle increment is 120°.

Example

(9). Number: $z \;=\; 3 \;+\; 4\,j \;=\; 5 \angle 53.13°$ [Three cube roots]

Root 1: $$z^{1/3} \;=\; 5^{1/3} \angle \left(\frac{53.13°}{3} \right) \;=\; 1.71 \angle 17.7°$$

Root 2: $$z^{1/3} \;=\; 5^{1/3} \angle \left(\frac{53.13° + 360°}{3} \right) \;=\; 1.71 \angle \left(17.7° + 120° \right) \;=\; 1.71 \angle 137.7°$$

Root 3: $$z^{1/3} \;=\; 5^{1/3} \angle \left(\frac{53.13° + 2 \times 360°}{3} \right) \;=\; 1.71 \angle \left(17.7° + 240° \right) \;=\; 1.71 \angle 257.7°$$

For higher order roots, we can work out the angular increment $\left(\dfrac{360°}{n} \right)$ and generate the required number of roots from the first-calculated root.

Further Example

(10). Determine the square roots of j, expressing both in polar and rectangular forms

Number: $z \;=\; j \;=\; 0 \;+\; 1\,j \;=\; 1 \angle 90°$ [Two square roots]

Root 1: $$z^{1/2} \;=\; 1^{1/2} \angle \left(\frac{90°}{2} \right) \;=\; \underline{\underline{1 \angle 45°}}$$

Rectangular form: $1 \angle 45° \;\rightarrow\; \underline{\underline{0.707 \;+\; 0.707\;j}}$

Angular increment $= \dfrac{360°}{2} \;=\; 180°$

Root 2: $$z^{1/2} \;=\; 1 \angle (45° + 180°) \;=\; \underline{\underline{1 \angle 225°}}$$

Rectangular form: $1 \angle 225° \;\rightarrow\; \underline{\underline{-0.707 \;-\; 0.707\;j}}$

Aside: **The Polar Form of a Complex Number as an Exponential**

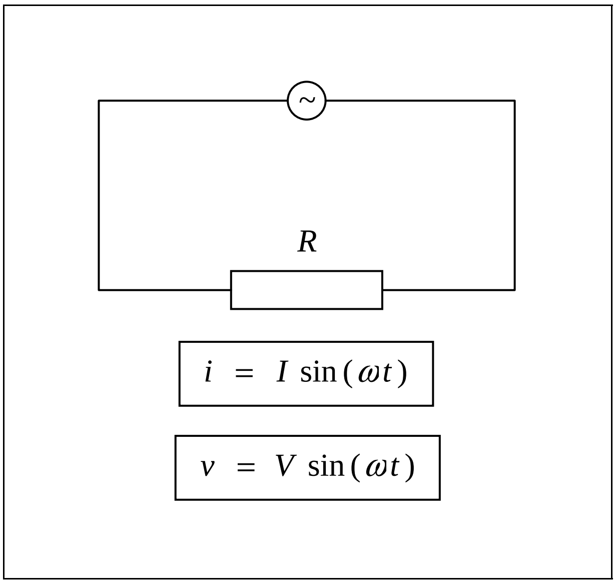
The arithmetic of complex numbers in polar form is reminiscent of the laws of indices. In fact it is entirely consistent within mathematics to represent the polar form of a complex number as a complex exponential. This gives rise to **Euler's formula**:

$$z = r(\cos\theta + j\sin\theta) = r\,e^{j\theta}.$$

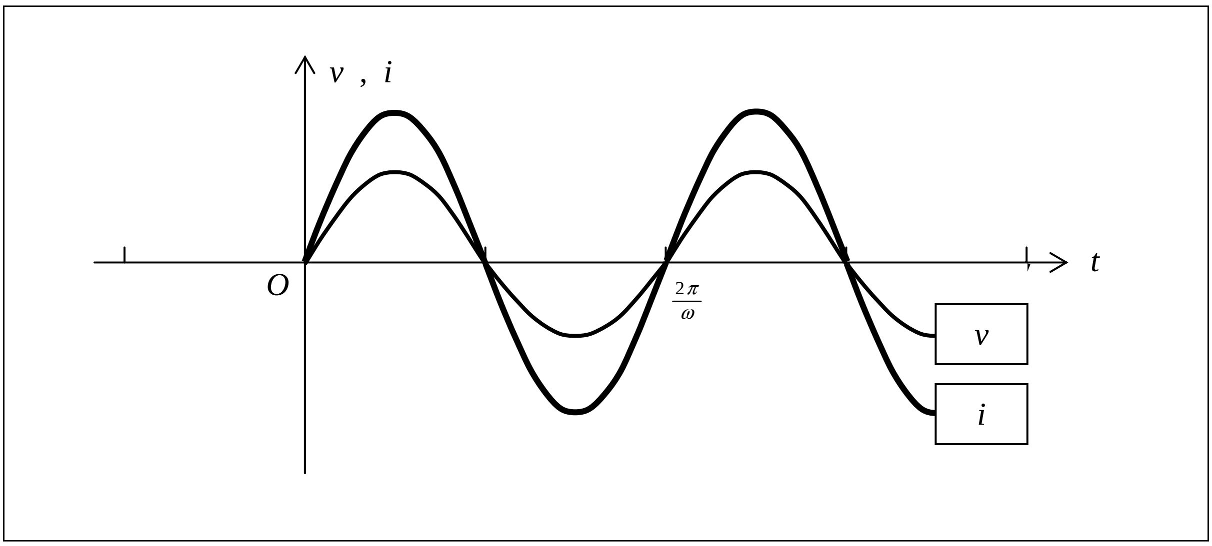
Algebraically, a complex exponential is handled just like an ordinary (real) one.

8) An Application of Complex Numbers to AC Circuits

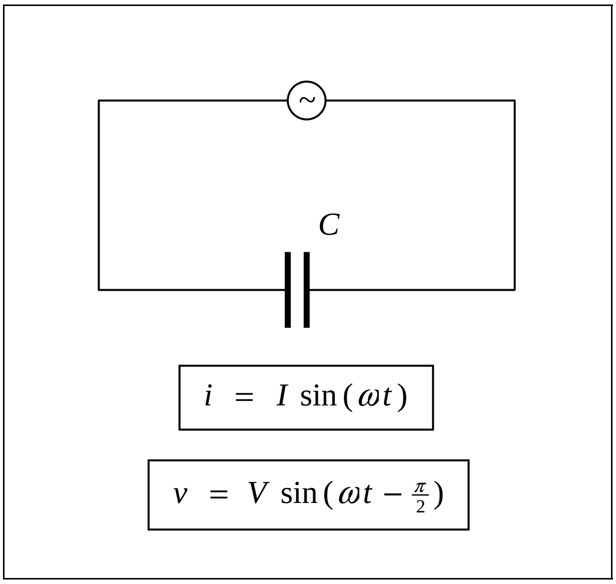
Consider a simple electronic circuit with an alternating source voltage and a single resistor with resistance R:



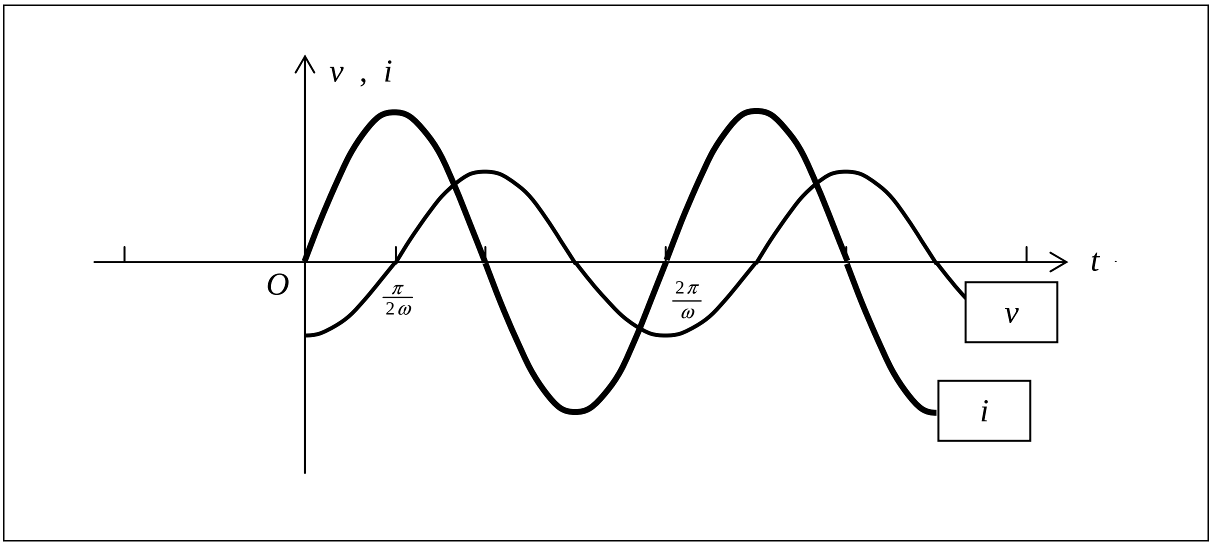
In this configuration, $\omega = 2\pi f$ where f is the frequency of the oscillation, $V = I R$ (from Ohm's law) and the voltage (v) and current (i) oscillate **in phase**:



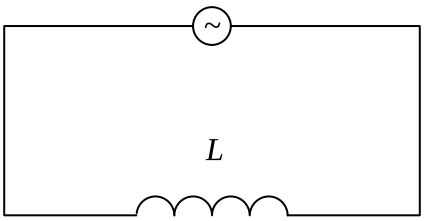
Now consider the case of a similar circuit, but with a capacitor (capacitance C) instead of a resistor:



In this case experimentation shows that $V = \dfrac{I}{\omega C}$ and the voltage (v) and current (i) oscillate **out of phase**, with the voltage “lagging” the current by $\frac{\pi}{2}$ (i.e. a relative phase angle of $-\frac{\pi}{2}$):

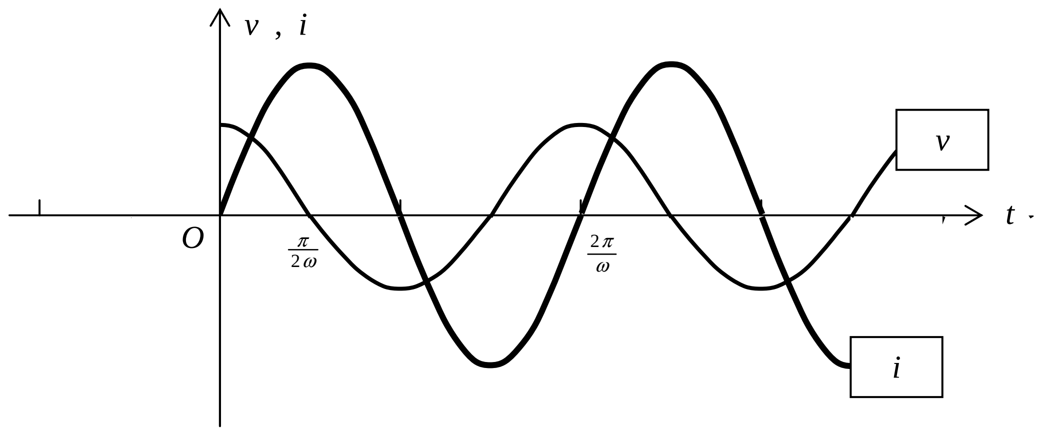


For a circuit with an inductor we find that $v = \omega L I$, where L is the inductance, and the voltage "leads" the current by $\frac{\pi}{2}$ (i.e. a relative phase angle of $+\frac{\pi}{2}$):



$$\boxed{i = I \sin(\omega t)}$$

$$\boxed{v = V \sin(\omega t + \tfrac{\pi}{2})}$$



The relative phase between voltage and current is important in circuit design. When components are combined it would be useful if the effects on relative phase could be calculated. Complex numbers provide the means.

Associated with each of the three types of components is a **complex impedance**. This is a complex number expressible either in rectangular or polar form:

Component	**Complex Impedance**
Resistor	$z_R = R + 0j = R \angle 0°$
Capacitor	$z_C = 0 - \dfrac{1}{\omega C} j = \dfrac{1}{\omega C} \angle(-90°)$
Inductor	$z_L = 0 + \omega L j = \omega L \angle(+90°)$

Note: The angles in the complex impedances are the same as the voltage phase angles observed on pages 15-17. Technically, we should express the angles in radians. However we shall be combining these impedances using complex arithmetic and you may find that a little easier to do by working in degrees.

Just to remind you:
R is resistance
C is capacitance
L is inductance
$\omega = 2\pi f$, where f is frequency .

The complex impedances combine in exactly the same way as resistances, but using complex arithmetic. In fact, the magnitude of an impedance is measured in ohms. Suppose we have two components with impedances z_1 and z_2.

Two components in series: Combined impedance : $z = z_1 + z_2$

Two components in parallel: Combined impedance : $\dfrac{1}{z} = \dfrac{1}{z_1} + \dfrac{1}{z_2}$

$$\text{or} \quad z = \frac{z_1 \, z_2}{z_1 + z_2}$$

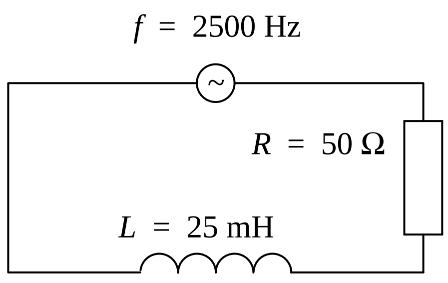
A complex version of Ohm's law relates voltage, current and impedance:

$$v = z.i \; .$$

In this formula, voltage and current are also complex numbers. It turns out that the way the angles of the polar forms of the complex numbers combine under the rules of complex arithmetic models the effect of the components on relative phase. Because of this, in this context, the complex numbers are sometimes called **phasors**.

Examples

(11). Determine the combined impedance of the following configuration of components:



$$\omega \;=\; 2\pi f \;=\; 2\pi \times 2500 \;=\; 5000\pi \;\approx\; 15707.96$$

$$z_R \;=\; \begin{cases} R \;+\; 0\,j \\ R \angle 0^\circ \end{cases} \;=\; \begin{cases} 50 \;+\; 0\,j \\ 50 \angle 0^\circ \end{cases}$$

$$\omega L \;=\; 5000\pi \times (25 \times 10^{-3}) \;\approx\; 392.699$$

$$z_L \;=\; \begin{cases} 0 \;+\; \omega L\,j \\ \omega L \angle 90^\circ \end{cases} \;=\; \begin{cases} 0 \;+\; 392.699\,j \\ 392.699 \angle 90^\circ \end{cases}$$

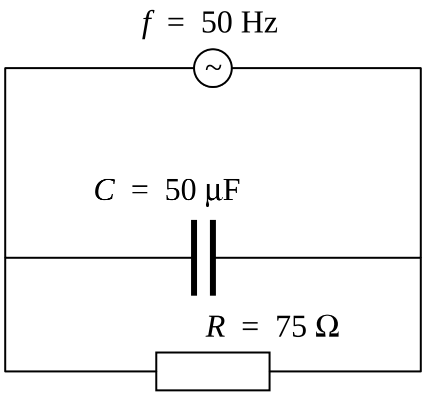
Components are in series so add (in rectangular form) to give combined impedance:

$$z_T \;=\; z_R \;+\; z_L \;=\; 50 \;+\; 392.699\,j\ .$$

Convert to polar form using a calculator:

$$\underline{\underline{z_T \;=\; 395.869 \angle 82.744^\circ}}\ .$$

(12). Determine the combined impedance of the following configuration of components:



$$\omega = 2\pi f = 2\pi \times 50 = 100\pi \approx 314.159$$

$$z_R = \begin{cases} R + 0j \\ R \angle 0° \end{cases} = \begin{cases} 75 + 0j \\ 75 \angle 0° \end{cases}$$

$$\frac{1}{\omega C} = \frac{1}{100\pi \times (50 \times 10^{-6})} \approx 63.662$$

$$z_C = \begin{cases} 0 - \dfrac{1}{\omega C} j \\ \dfrac{1}{\omega C} \angle (-90°) \end{cases} = \begin{cases} 0 - 63.662\, j \\ 63.662 \angle (-90°) \end{cases}$$

Components are in parallel so the combined impedance is given by: $z_T = \dfrac{z_R\ z_C}{z_R + z_C}$.

Upper product best evaluated in polar form:

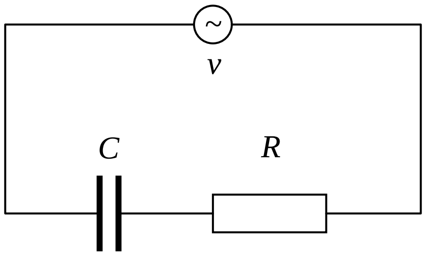
$$\begin{aligned} z_R\ z_C &= (75 \angle 0°)\,(63.662 \angle (-90°)) \\ &= 4774.65 \angle (-90°) \end{aligned}$$

Lower sum evaluated in rectangular form, then converted to polar form before completing final division in polar form:

$$\begin{aligned} z_R + z_C &= (75 + 0\,j) + (0 - 63.662\,j) \\ &= 75 - 63.662\,j \\ &= 98.38 \angle (-40.33°) \end{aligned}$$

$$z_T = \frac{z_R\ z_C}{z_R + z_C} = \frac{4774.65 \angle (-90°)}{98.38 \angle (-40.33°)} = \underline{\underline{48.53 \angle (-49.67°)}}\ .$$

(13). Consider the following circuit with two components in series:



Suppose that: the source voltage is 5 mV at a frequency of 1000 Hz ;

the resistance of the resistor is 30 Ω ;

the capacitance of the capacitor is 10 μF .

Determine:

(i). the circular frequency ω ;

(ii). the total impedance of the circuit components;

(iii). the current i in complex polar form (assuming a zero phase for the source voltage);

(iv). the voltage across each component.

(i). $\omega = 2\pi f = 2\pi \times 1000 = 2000\pi \approx \underline{\underline{6283.185}}$

(ii). Impedance of R : $z_R = 30 + 0j = 30 \angle 0°$

Impedance of C : $\dfrac{1}{\omega C} = \dfrac{1}{2000\pi \times 10 \times 10^{-6}} = \dfrac{50}{\pi} \approx 15.915$

$z_C = 0 - 15.915j = 15.915 \angle (-90°)$

Combined impedance : Components in series, so add impedances (in rectangular form)

$$\begin{aligned} z_T &= z_R + z_C \\ &= 30 - 15.915j \\ &= \underline{\underline{33.96 \angle (-27.946°)}} \end{aligned}$$

(iii). Set the complex form of the voltage: $v = 0.005 \angle 0°$

By Ohm's law, the current is given by:

$$\begin{aligned} i &= \frac{v}{z_T} \\ &= \frac{0.005 \angle 0°}{33.96 \angle (-27.946°)} \\ &= \underline{\underline{(1.472 \times 10^{-4}) \angle (+27.946°)}} \end{aligned}$$

(iv). Voltage across the resistor:

$$\begin{aligned} v_R &= z_R \; i \\ &= (30 \angle 0°)\,(1.472 \times 10^{-4} \angle 27.946°) \\ &= \underline{\underline{4.416 \times 10^{-3} \angle 27.946°}} \end{aligned}$$

Voltage across the capacitor:

$$\begin{aligned} v_C &= z_C \; i \\ &= (15.915 \angle (-90°))\,(1.472 \times 10^{-4} \angle 27.946°) \\ &= \underline{\underline{2.343 \times 10^{-3} \angle (-62.054°)}} \end{aligned}$$

Tutorial Exercises

Q1. Express each of the following expressions in the Cartesian (i.e. rectangular) complex form $a + bj$:

(i). $2 - \sqrt{-4}$ **(ii).** $-8 + \sqrt{-25}$

(iii). $-6 + \sqrt{-16}$ **(iv).** $5 + \sqrt{-12}$.

Q2. Determine the complex solutions of the following equations:

(i). $z^2 + 36 = 0$ **(ii).** $z^2 + 27 = 0$

(iii). $z^2 + 8z + 20 = 0$ **(iv).** $z^2 + z + 1 = 0$.

Q3. Simplify the following complex expressions, expressing each in the form $a + bj$:

(i). $(4 + 11j) + (8 + 6j)$ **(ii).** $(8 + j) - (4 + 3j)$

(iii). $(-8 - 4j) + (3 + 6j)$ **(iv).** $(9 + 5j) - (-1 - 8j)$

(v). $(2 + 5j)(4 + 2j)$ **(vi).** $(8 - 7j)(7 + 8j)$

(vii). $(-3 + 4j)(6 - 3j)$ **(viii).** $(4 - 3j)(4 + 3j)$

(ix). $(3 + 6j)^2$ **(x).** $(2 - 2j)^2$

(xi). $(-5 + 4j)^2$ **(xii).** $(-8 - 3j)^2$

(xiii). j^8 [Hint: $= (j^2)^4$] **(xiv).** j^{10} [Hint: $= (j^2)^5$]

(xv). j^9 [Hint: $= j^8 . j$] **(xvi).** j^{11} [Hint: $= j^{10} . j$]

(xvii). $(1 + 2j)^3$ **(xviii).** $(p + qj)^2$.

Q4. Simplify the following complex divisions to rectangular form:

(i). $\dfrac{3 + 2j}{1 - 2j}$ **(ii).** $\dfrac{4 - 5j}{4 + 6j}$

(iii). $\dfrac{8 - j}{4 + j}$ **(iv).** $\dfrac{9 + 2j}{-1 - j}$

(v). $\dfrac{1}{4 + 3j}$ **(vi).** $\dfrac{1}{12 - 5j}$.

Q5. Graph each of the following complex numbers on an Argand diagram (i.e. an *Oxy* axes system) and, **<u>without the aid of a calculator</u>**, express each in polar form:

(i). $2 + 2j$ **(ii).** $-4 + 4j$

(iii). $-3 - 3j$ **(iv).** $5 - 5j$

(v). 8 [Hint: $= 8 + 0j$] **(vi).** 25

(vii). -4 **(viii).** -7

(ix). $2j$ [Hint: $= 0 + 2j$] **(x).** $6j$

(xi). $-5j$ **(xii).** $-20j$.

Q6. Convert each of the following complex numbers to polar form using both conversion formulae and your calculator's conversion facility:

(i). $3 + 4j$ **(ii).** $-3 + 4j$

(iii). $-5 + 4j$ **(iv).** $-5 - 4j$.

[**Note:** When using conversion formulae, remember to use a sketch to establish the correct quadrant for the angle.]

Q7. Four complex numbers in polar form are

$$z_1 = 4 \angle 30^\circ \qquad z_2 = 3 \angle (-50^\circ)$$

$$z_3 = 2 \angle 120^\circ \qquad z_4 = 6 \angle (-100^\circ) .$$

(i). Sketch each complex number on an Argand diagram.

(ii). Determine the products $z_1 z_3$, $z_3 z_4$ and $z_2 z_4$ in their polar forms.

(iii). Determine the quotients $\dfrac{z_1}{z_3}$, $\dfrac{z_3}{z_1}$, $\dfrac{z_2}{z_4}$ and $\dfrac{z_4}{z_1}$ in their polar forms.

(iv). Determine $\dfrac{z_2 z_3}{z_1 z_4}$ in polar form.

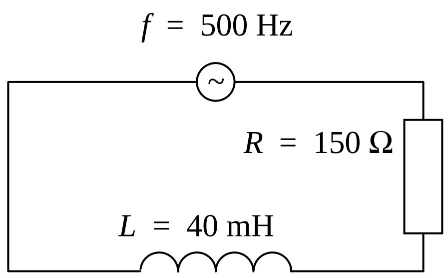
(v). Convert z_1 , z_2 , z_3 and z_4 to their rectangular forms using both conversion formulae and your calculator's conversion facility.

Q8. (i). Determine the square roots of $4 \angle 60^\circ$.

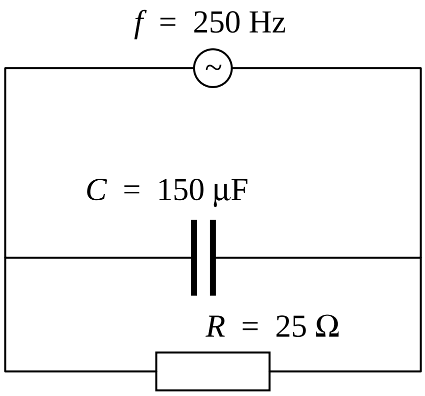
(ii). Determine the square roots of $1 - j$ in polar form.

(iii). Determine the cube roots of j in polar and rectangular forms.

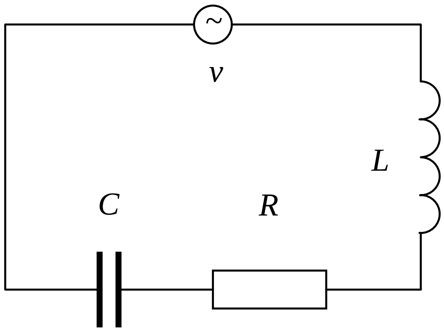
Q9. Determine the combined impedance of the following configuration of components:



Q10. Determine the combined impedance of the following configuration of components:



Q11. Consider the following circuit with two components in series:



Suppose that: the source voltage is 8 mV at a frequency of 750 Hz ;

the resistance of the resistor is 50 Ω ;

the inductance of the inductor is 250 mH ;

the capacitance of the capacitor is 20 μF .

Determine:

(i). the circular frequency ω ;

(ii). the total impedance of the circuit components;

(iii). the current i in complex polar form (assuming a zero phase for the source voltage);

(iv). the voltage across each component.

Answers

A1. **(i).** $2 - 2j$ **(ii).** $-8 + 5j$

(iii). $-6 + 4j$ **(iv).** $5 + 2\sqrt{3}\,j$.

A2. **(i).** $z = \pm 6j$ **(ii).** $z = \pm 3\sqrt{3}\,j$

(iii). $z = -4 \pm 2j$ **(iv).** $z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}j$.

A3. **(i).** $12 + 17j$ **(ii).** $4 - 2j$

(iii). $-5 + 2j$ **(iv).** $10 + 13j$.

(v). $-2 + 24j$ **(vi).** $112 + 15j$

(vii). $-6 + 33j$ **(viii).** 25 or $25 + 0j$

(ix). $-27 + 36j$ **(x).** $-8j$ or $0 - 8j$

(xi). $9 - 40j$ **(xii).** $55 + 48j$

(xiii). $+1$ **(xiv).** -1

(xv). j **(xvi).** $-j$

(xvii). $-11 - 2j$ **(xviii).** $(p^2 - q^2) + 2pqj$.

A3. **(i).** $-\frac{1}{5} + \frac{8}{5}j$ **(ii).** $-\frac{7}{26} - \frac{11}{13}j$

(iii). $\frac{31}{17} - \frac{12}{17}j$ **(iv).** $-\frac{11}{2} + \frac{7}{2}j$

(v). $\frac{4}{25} - \frac{3}{25}j$ **(vi).** $\frac{12}{169} + \frac{5}{169}j$.

A5. **(i).** $2\sqrt{2} \angle 45°$ **(ii).** $4\sqrt{2} \angle 135°$

(iii). $3\sqrt{2} \angle 225°$ or $3\sqrt{2} \angle (-135°)$

(iv). $5\sqrt{2} \angle 315°$ or $5\sqrt{2} \angle (-45°)$

(v). $8 \angle 0°$ **(vi).** $25 \angle 0°$

(vii). $4 \angle 180°$ **(viii).** $7 \angle 180°$

(ix). $2 \angle 90°$ **(x).** $6 \angle 90°$

(xi). $5 \angle 270°$ or $5 \angle (-90°)$ **(xii).** $20 \angle 270°$ or $20 \angle (-90°)$.

A6. **(i).** $5 \angle 53.13°$ **(ii).** $5 \angle 126.87°$

(iii). $6.403 \angle 141.34°$ **(iv).** $6.403 \angle (-141.34°)$.

A6. **(ii).** $z_1\, z_3 \;=\; 8 \angle 150°$

$z_3\, z_4 \;=\; 12 \angle 20°$

$z_2\, z_4 \;=\; 18 \angle (-150°)$

(iii). $\dfrac{z_1}{z_3} \;=\; 2 \angle (-90°)$

$\dfrac{z_3}{z_1} \;=\; 0.5 \angle 90°$

$\dfrac{z_2}{z_4} \;=\; 0.5 \angle 50°$

$\dfrac{z_4}{z_1} \;=\; 1.5 \angle (-130°)$

(iv). $\dfrac{z_2\, z_3}{z_1\, z_4} \;=\; 0.25 \angle 140°$

(v). $z_1 \;=\; 4 \angle 30° \;\rightarrow\; 3.464 \;+\; 2\, j$

$z_2 \;=\; 3 \angle (-50°) \;\rightarrow\; 1.928 \;-\; 2.298\, j$

$z_3 \;=\; 2 \angle 120° \;\rightarrow\; -1 \;+\; 1.732\, j$

$z_4 \;=\; 6 \angle (-100°) \;\rightarrow\; -1.042 \;-\; 5.909\, j$.

A8. **(i).** $2 \angle 30° \quad , \quad 2 \angle 210°$

(ii). $1.189 \angle(-22.5°) \quad , \quad 1.189 \angle 157.5°$

(iii). $1 \angle 30° \;\rightarrow\; 0.866 \;+\; 0.5\, j$

$1 \angle 150° \;\rightarrow\; -0.866 \;+\; 0.5\, j$

$1 \angle 270° \;\rightarrow\; 0 \;-\; j \;=\; -j$.

A9. $\omega = 2\pi f = 2\pi \times 500 = 1000\pi \approx 3141.59$

$$z_R = \begin{cases} R + 0j \\ R \angle 0^\circ \end{cases} = \begin{cases} 150 + 0j \\ 150 \angle 0^\circ \end{cases}$$

$$\omega L = 1000\pi \times (40 \times 10^{-3}) \approx 125.664$$

$$z_L = \begin{cases} 0 + \omega L j \\ \omega L \angle 90^\circ \end{cases} = \begin{cases} 0 + 125.664j \\ 125.664 \angle 90^\circ \end{cases}$$

Components are in series so add (in rectangular form) to give combined impedance:

$$z_T = z_R + z_L = 150 + 125.664j \ .$$

Convert to polar form using a calculator:

$$z_T = 195.682 \angle 39.955^\circ \ .$$

A10. $\omega = 2\pi f = 2\pi \times 250 = 500\pi \approx 1570.80$

$$z_R = \begin{cases} R + 0\,j \\ R \angle 0° \end{cases} = \begin{cases} 25 + 0\,j \\ 25 \angle 0° \end{cases}$$

$$\frac{1}{\omega C} = \frac{1}{500\pi \times (150 \times 10^{-6})} \approx 4.244$$

$$z_C = \begin{cases} 0 - \dfrac{1}{\omega C}\,j \\ \dfrac{1}{\omega C} \angle (-90°) \end{cases} = \begin{cases} 0 - 4.244\,j \\ 4.244 \angle (-90°) \end{cases}$$

Components are in parallel so the combined impedance is given by: $z_T = \dfrac{z_R\ z_C}{z_R + z_C}$.

Upper product best evaluated in polar form:

$$\begin{aligned} z_R\ z_C &= (25 \angle 0°)\,(4.244 \angle (-90°)) \\ &= 106.10 \angle (-90°) \end{aligned}$$

Lower sum evaluated in rectangular form, then converted to polar form before completing final division in polar form:

$$\begin{aligned} z_R + z_C &= (25 + 0\,j) + (0 - 4.244\,j) \\ &= 25 - 4.244\,j \\ &= 25.358 \angle (-9.63°) \end{aligned}$$

$$z_T = \frac{z_R\ z_C}{z_R + z_C} = \frac{106.10 \angle (-90°)}{25.358 \angle (-9.63°)} = 4.18 \angle (-80.37°) \quad .$$

A11. (i). $\omega = 2\pi f = 2\pi \times 750 = 1500\pi \approx 4712.39$

(ii). Impedance of R : $z_R = 50 + 0j = 50 \angle 0°$

Impedance of L : $\omega L = 1500\pi \times (250 \times 10^{-3}) \approx 1178.097$

$$z_L = \begin{cases} 0 + \omega L j \\ \omega L \angle 90° \end{cases} = \begin{cases} 0 + 1178.097j \\ 1178.097 \angle 90° \end{cases}$$

Impedance of C : $\dfrac{1}{\omega C} = \dfrac{1}{1500\pi \times 20 \times 10^{-6}} \approx 10.610$

$$z_C = 0 - 10.610j = 10.610 \angle (-90°)$$

Combined impedance : Components in series, so add impedances (in rectangular form)

$$\begin{aligned} z_T &= z_R + z_L + z_C \\ &= 50 + 1167.487j \\ &= 1168.557 \angle 87.548° \end{aligned}$$

(iii). Set the complex form of the voltage: $v = 0.008 \angle 0°$

By Ohm's law, the current is given by:

$$\begin{aligned} i &= \frac{v}{z_T} \\ &= \frac{0.008 \angle 0°}{1168.557 \angle 87.548°} \\ &= (6.846 \times 10^{-6}) \angle (-87.548°) \end{aligned}$$

(iv). Voltage across the resistor:

$$\begin{aligned} v_R &= z_R \, i \\ &= (50 \angle 0°)\,(6.846 \times 10^{-6} \angle (-87.548°)) \\ &= 3.423 \times 10^{-4} \angle (-87.548°) \end{aligned}$$

Voltage across the inductor:

$$
\begin{aligned}
v_L &= z_L \; i \\
&= (1178.097 \angle 90^\circ)\,(6.846 \times 10^{-6} \angle (-87.548^\circ)) \\
&= 8.065 \times 10^{-3} \angle 2.452^\circ
\end{aligned}
$$

Voltage across the capacitor:

$$
\begin{aligned}
v_C &= z_C \; i \\
&= (10.610 \angle (-90^\circ))\,(6.846 \times 10^{-6} \angle (-87.548^\circ)) \\
&= 7.264 \times 10^{-5} \angle (-177.548^\circ)
\end{aligned}
$$